B.Sc. IV SEMESTER

Mathematics

PAPER – II

GROUP THEORY, FOURIER SERIES
AND
DIFFERENTIAL EQUATIONS

UNIT-I

GROUP THEORY-III

Syllabus:

Unit – I

Normal sub-groups, Quotient groups, Homomorphism and Isomorphism of groups, Kernel of Homomorphism, Fundamental theorem of Homomorphism.

-10HRS

10/01/2020

Group theory - III

Normal Subgroup : (Invariant Subgroup)

Let $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$

R.C

$H.1 = \{1, -1\} = H$

$H(-1) = \{-1, 1\} = H$

$H(i) = \{i, -i\} = H$

$H(-i) = \{-i, i\} = H$

L.C

$1.H = \{1, -1\} = H$

$(-1)H = \{-1, +1\} = H$

$i(H) = \{i, -i\} = H$

$(-i)(H) = \{-i, i\} = H.$

$\therefore$ RC = L.C.

if $Ha = aH \quad \forall a \in G.$

$h_1 a = a h_2$

multiply by a^{-1} $\Rightarrow h_1 a(a^{-1}) = (ah_2)a^{-1}$

$h_1(aa^{-1}) = a h_2 a^{-1}$

$h_1(e) = a h_2 a^{-1} \Rightarrow h_1 = a h_2 a^{-1}$

A $H \triangle G$, $\forall g \in G$ and $h \in H$.

$ghg^{-1} \in H.$

$ghg^{-1} = h$

$(ghg^{-1})g = hg \Rightarrow \because$ multiplied by g.

$gh(g^{-1}g) = hg$

$\rightarrow gh = hg$

$\rightarrow gH = Hg \quad \forall h \in H.$

Defination of Normal Sub group : A Subgroup H of group G is said to be normal subgroup of G, if $ghg^{-1} \in H \quad \forall g \in G$ and $h \in H$, denoted by $H \triangle G$ and read as H is normal in G.

Note

* A Set containing $\{e\}$ and a Group G itself are always normal Subgroup in G. and are called trivial or improper normal subgroup.

→ Any other normal subgroup of G other than these two are called proper or non trivial normal subgroup.

* Every subgroup of an abelian group is normal.
ex : $(n\mathbb{Z}, +)$ is normal subgroup of $(\mathbb{Z}, +)$ because $(\mathbb{Z}, +)$ is an abelian group.

* Every Subgroup of a cyclic group is normal.

* But subgroup of non abelian group is may be normal.

Theorem 1 : A Subgroup H of a group G is normal iff $gHg^{-1} = H \ \forall \ g \in G$.

⟹ Let H be a normal subgroup of G.
i.e $H \Delta G$

$gHg^{-1} = \{ghg^{-1} / h \in H\}$ [By defination]

Let $x \in gHg^{-1}$

$x = ghg^{-1} \quad \forall \ h \in H.$

$x \in H \ (\because H \Delta G)$

→ $gHg^{-1} \subset H \longrightarrow (1)$

Replacing 'g' by g^{-1}

$g^{-1}H(g^{-1})^{-1} \subseteq H$

$g^{-1}Hg \subseteq H$

$H = (gg^{-1})H(gg^{-1})$ → (By defination. $H = gHg^{-1}$)

$H = gHg^{-1}$

$$H \subseteq gHg^{-1} \longrightarrow ②$$

from ① and ②.

$$gHg^{-1} = H.$$

Conversely: $gHg^{-1} = H \quad \forall\, g \in G.$

$ghg^{-1} \subset gHg^{-1} \quad \forall\, h \in H.$

By data.

$$gHg^{-1} = H \quad \forall\, g \in G.$$

$$ghg^{-1} \in H \quad \forall\, h \in G \text{ and } g \in G.$$

Hence $H \Delta G.$

Theorem 2: A subgroup H of a group G is normal iff every right coset of H in G is a left coset of H in G.

$\Rightarrow$ Let H be a normal subgroup of G.

by theorem 1, $gHg^{-1} = H \quad \forall\, g \in G.$

$$(gHg^{-1})g = Hg \quad \forall\, g \in G.$$

$$(gH)(g^{-1}g) = Hg \quad \forall\, g \in G.$$

$$gH = Hg \quad \forall\, g \in G.$$

$\therefore$ Every right coset of H in G is a left coset of H in G.

conversely,

let every right coset H in G be equal to left coset of H in G.

Let the right coset $Hg \quad \forall\, g \in G$ be equal to left coset say $xH \quad \forall\, x \in G$.

Now $g \in Hg = xH$ Also $g \in gH.$

$g \in xH$ & $g \in gH,$

$\Rightarrow gH = xH$ ($\because$ any two left cosets are either identical or closure).

$\Rightarrow gH = Hg \quad \forall g \in G.$

$gHg^{-1} = H.$

$\therefore H \trianglelefteq G.$

Theorem III : A Subgroup H of a group G is normal iff, the product of any two right cosets or (left cosets) of H in G is ~~given~~ again a right (left coset) of H in G.

$\Rightarrow$ Let H be a normal in G.
and Ha and Hb be any two right coset of H.

Consider $Ha \cdot Hb = H(aH)b.$

$= H(Ha)b \quad (\because H$ is normal $Ha = aH)$

$= (HH)ab$

$= H \cdot ab \quad (\because H$ is a subgroup $H \cdot H = H)$

$\therefore$ The product of 2 right cosets of H in G is again a right cosets of H in G.

Conversely, Let the product of any 2 right cosets of H in G be again a right coset.
Hx & Hx^{-1} will be two right coset $\forall x \in G$.

By data, $Hx \cdot Hx^{-1}$ is again a right cosets

$\therefore x \in Hx$ & $x^{-1} \in Hx^{-1}$.

$e = xx^{-1} \in Hx \cdot Hx^{-1}$

$\therefore e \in H$ which is right coset of H.

Two right cosets, H & $Hx \cdot Hx^{-1}$ has an element in common.

$\therefore Hx \cdot Hx^{-1} = H$ (Since two right cosets of H are either equal or disjoint).

$Hx \cdot Hx^{-1} = H.$

$\Rightarrow h_1 x \cdot h_2 x^{-1} \in H \quad \forall x \in G,\ h_1, h_2 \in H.$

$\Rightarrow x h_2 x^{-1} \in (h_1)^{-1} H \quad \forall x \in G,\ h_1, h_2 \in H.$

$\Rightarrow x h_2 x^{-1} \in H$

$\Rightarrow H \trianglelefteq G.$

Results ;

1) The intersection of any 2 normal subgroups of a group is also a normal subgroup.

$\Rightarrow$ Let H and K be any 2 normal subgroups of G. Since H and K are subgroups.

$H \cap K$ is also a subgroup of G.

Let $x \in G$ and $h \in H \cap K$ be any arbitrary.

$\therefore x \in G$ and $h \in H$, $h \in K$.

$\therefore x h x^{-1} \in H$ & $x h x^{-1} \in K$.

($\because H \trianglelefteq G$ and $K \trianglelefteq G$)

$\Rightarrow x h x^{-1} \in H \cap K \quad \forall x \in G$ & $h \in H \cap K$.

$\Rightarrow$ Hence $H \cap K$ is normal in G.

2) The product of any two normal subgroups of a group is a subgroup of the group.

$\Rightarrow$ Let H and K be any two normal subgroups of the group G.

$$HK = \{ h_1 k_1 \mid \forall h_1 \in H,\ k_1 \in K \}$$

Since $e \in H$ and $e \in K$.

$e \cdot e = e \in HK.$

HK is non empty.

Let $x, y \in H \cdot K$ be arbitary.

Consider,

$$xy^{-1} = (h_1 k_1)(h_2 k_2)^{-1} \quad \forall\ h_1, h_2 \in H,\ k_1, k_2 \in K$$
$$= h_1 (h_2^{-1} h_2)(k_1 k_2^{-1} h_2^{-1})$$

$$= (h_1 h_2^{-1}) \left[h_2 (k_1 k_2^{-1}) h_2^{-1} \right]$$
$$= h \cdot k \quad \left(\because h = h_1 h_2^{-1} \right.$$
$$k = h_2 (k_1 k_2^{-1})(h_2^{-1}).$$

Now, $h_2 \in H \Longrightarrow h_2 \in G$, and $k_1 k_2^{-1} \in K$.

$$h_2 (k_1 k_2^{-1}) h_2^{-1} \in K.$$

$\Rightarrow k \in K$

$\therefore xy^{-1} = h \cdot k \in HK.$

Hence HK is a subgroup of G.

3) If H ~~be~~ is a subgroup and K is a normal subgroup of a group G. then ~~R~~ $H \cap K$ is normal in H.

$\Rightarrow$ (Let H be a subgroup and K ~~be~~ a normal subgroup)

$\rightarrow$ Since H and K are subgroup of G then, $H \cap K$ is also a subgroup of G.

$$H \cap K \subset H$$

$\therefore H \cap K$ is a subgroup of H.

Let $g \in H$ and $h \in H \cap K$ be arbitary.

$h \in H \cap K \Rightarrow h \in H$ and $h \in K$.

~~K and~~ $K \triangle H \Rightarrow ghg^{-1} \in K \longrightarrow (1)$

Consider,

$hg^{-1} \in K$ as H is a sub group.

$ghg^{-1} \in H \longrightarrow (\because g \in H,\ hg^{-1} \in H)$

From ① and ②.

$ghg^{-1} \in H \cap K$.

$\forall g \in H$ and $h \in H \cap K$.

Hence, $H \cap K$ is normal in H.

Note :

* The Centre Z of a group G is set of those elements of G which commute with every element of G.

$$Z = \{ z \in G \mid zx = xz \quad \forall x \in G \}$$

4) The Centre Z of Group G is a normal subgroup of G.

Z is a subgroup of G.

To show that : Z is normal in G.

Let $x \in G$ and $z \in Z$ be arbitary.

$$\begin{aligned} xzx^{-1} &= (xz)x^{-1} \\ &= (zx)x^{-1} \quad (\because z \in Z) \\ &= z(xx^{-1}) \\ &= z. \end{aligned}$$

$\therefore$ $xzx^{-1} = z \in Z \quad \forall x \in G$ and $z \in G$.

Hence Z is normal subgroup of G.

Quotient Group :

Let G be a group, H be a subgroup of G then the set $\frac{G}{H} = \{ Ha \mid a \in G \}$ forms a group under coset multiplication $Ha \cdot Hb = Hab$.

this group $\left(\frac{G}{H}\right)$ is called a quotient group or factor group of H in G.

Here the subgroup H itself acts as identity element of $\frac{G}{H}$

it means $\forall\ Ha \in \frac{G}{H} \Rightarrow Ha^{-1} \in \frac{G}{H}$ is inverse element.

Note : If the binary operation is addition then. $\frac{G}{H} = \{ H+a \,/\, a \in G \}$ form a group under coset addition,

$$(H+a) + (H+b) = H + (a+b)$$

The identity is $H+0$.

and inverse is $H-a \in \frac{G}{H}$ is inverse element.

If G is abelian $\frac{G}{H}$ is abelian but, converse is not true.

i.e if $\frac{G}{H}$ is abelian then G may or may not be abelian.

* If H is a normal subgroup of finite order then,

$$o\left(\frac{G}{H}\right) = \frac{o(G)}{o(H)}$$

$\Rightarrow$ By defination : $o\left(\frac{G}{H}\right)$ = no of distinct cosets of H

= Index of H in G.

$$o\left(\frac{G}{H}\right) = \frac{o(G)}{o(H)}$$ (By langranges theorem)

Ex : 1) If G is a group and H is a subgroup of index 2 in G. show that H is normal subgroup of G.

$\Rightarrow$ Since the index of H in G is 2 there are only two left (right) cosets.

let $x \in G$ be arbitary.

If $x \in H$, then $Hx = H$ and $xH = H$.

$Hx = xH$.

$\Rightarrow H$ is normal in G.

Supposing $x \notin H$.

$\Rightarrow Hx \neq H$ and $xH \neq H$.

Since the index of H is 2

the cosets H, Hx and xH are

$G = H \cup Hx = H \cup Hx$.

and $\phi = H \cap Hx = H \cap xH$.

$\therefore Hx = xH \quad \forall x \in G$.

Hence H is normal subgroup of G.

(2) Let H and K be two normal subgroups of a group G such that,

$H \cup K = \{e\}$ where; $e \rightarrow$ identity element of G.

Show that; every $h \in H$, $k \in H$, $hk = kh$

$\Rightarrow$ We shall show that

$hkh^{-1}k^{-1} = e$.

$\Rightarrow hk = kh$.

Since H is normal,

$khk^{-1} \in H \quad \forall\ h \in H$ and $k \in K \subset G$

$h \in H$ and $kh^{-1}k^{-1} \in H$.

$\Rightarrow hkh^{-1}k^{-1} \in H$. [$\because H$ is Subgroup]

Again K is normal.

$hkh^{-1} \in K \quad \forall\ k \in K$ and $h \in H \subset G$

Now,

$k^{-1} \in K$ and $hkh^{-1} \in K$.

$\Rightarrow hkh^{-1}k^{-1} \in K$ [Since K is subgroup]

$hkh^{-1}k^{-1} \in H \cap K = \{e\}$ [By Data]

$\Rightarrow hkh^{-1}k^{-1} = e \Rightarrow hk = kh$.

$\forall\ h \in H$ and $k \in K$.

③ ST every quotient group of an abelian group is abelian.

$\Rightarrow$ Let G be abelian group and N be a normal Subgroup of G.

Let $Na, Nb \in \frac{G}{N}$ be arbitary.

$$(Na)(Nb) = Nab$$
$$= Nba$$
$$= Nb \cdot Na \qquad (\because ab = ba)$$

$\Rightarrow \frac{G}{N}$ is abelian.

4) S.T Every factor group of cyclic group is cyclic.

$\Rightarrow$ Let 'G' be a cyclic group with generator 'a' & 'N' be a normal subgroup of 'G'

We shall show that factor group $\frac{G}{N}$ is cyclic this group, we shall show that elements of $\frac{G}{N}$ are generated by co-set Na.

$$Na^m = (Na)^m \longrightarrow (1)$$

Case - I

Let $m = 0$.

then $Na^m = Na^0 = Ne = N$.

$(Na)^m = (Na)^0 = \text{Identity of } \frac{G}{N}$

$= N$

$Na^m = (Na)^m$.

Case - II

Let m be a +ve integer.

$Na^m = Na, a \ldots a = Na, Na \ldots Na$

$(\because Nab = Na.Nb)$

$= (Na)^m$.

Case - III

Let m be a negative integer.

Let $m = -n$

where $n \rightarrow$ is +ve integer.

$(Na)^m = Na^{-n} = N(a^{-1})^n$.

$= Na^{-1}.Na^{-1}.Na^{-1} \ldots Na^{-1}$

$= (Na^{-1})^n$

$= (a^{-1}N)^n \quad (\because N \Delta G)$

$= [(Na^{-1})]^n$

$= (Na)^{-n}$

$= (Na)^m$

Hence (1) is true in all cases

Let $Ng \in \frac{G}{N}$ be arbitary. $g \in G$.

$\Rightarrow g = a^m \quad \forall \; m \in \mathbb{Z}$.

$Ng = Na^m = (Na)^m$

$\therefore$ Every element of $\frac{G}{N}$ is Some power of Na.

$\therefore \frac{G}{N}$ is Cyclic with Na as generator.

Homomorphism and Isomorphism.

Homomorphism of a group, consider a maping defined from one group to another which preserves the binary operation. Such mapping are called Homomorphism.

If two groups are homomorphism then they have Similar structure.

Let $(G, \cdot)$ and $(G', *)$ be a two groups of mapping $F : G \rightarrow G'$ is Said to be homomorphism if $f(g_1 \cdot g_2) = f(g_1) * f(g_2) \quad \forall \; g_1 g_2 \in G$.

F is said to be homomorphism if f maps is the composition of any two elements of G in to the composition of image of elements. This f preserves the compositions if F is Homomorphism from the group G into G' we say that G is homomorphic to group G'.

1) If the operation in both the group is multiplication then the above condition becomes $g(a \cdot b) = g(a) \cdot g(b)$

2) If the operation in both the group is addition then the above condition becomes,

$$g(a+b) = g(a) + g(b).$$

ISOMORPHISM :

A Homomorphism 'f' from a group G into G' is said to isomorphism if f is bijection (f is both one-one and on to).

Two groups G and G' are said to be isomorphic if $\exists$ isomorphism $f : G \to G'$, then we denote it by $G \cong G'$

* Endomorphism :

A Homomorphism 'f' from a group G on to itself is called endomorphism.

* Automorphism :

An isomorphism 'f' from a group G onto itself is called automorphism.

Ex: i) If $\phi : G \to G$, If ϕ is homomorphism G into G then ϕ is endomorphism.

Sol^n $\Rightarrow$ Let $\phi : G \to G'$ be defined by
$\phi(x) = e'$ $\forall x \in G$. (trivial mapping)

clearly ϕ is homomorphism.
let $a, b \in G$ be arbitary.
$\phi(a) = e'$, $\phi(b) = e'$. $\forall a, b \in G$

Ex: 2) If $(Z, +)$ i.e (Group of integers under addition) and if $f(x) = 2x$ $\forall x \in G$.
P.T f is homomorphism.

$\Rightarrow$ Here $f(x) = 2x$, $\forall x \in G$

$\forall x \in G \Rightarrow x+y \in G$.

$f(x) = 2x$; $f(y) = 2y$.

$$f(x+y) = 2(x+\cancel{2}y)$$

$$= \cancel{2(x+y)}.\ 2x+2y.$$

$$f(x+y) = f(x)+f(y)$$

Ex :3) If $f:(R,+) \rightarrow (R^+, \cdot)$ defined by $f(x) = 2^x$ $\forall x \in R$. then verify f is homomorphism or not?

$\Rightarrow$ $f(x) = 2^x$, $f(y) = 2^y$, $x+y \in R$.

$$f(x+y) = 2^{x+y}$$

$$= 2^x \cdot 2^y$$

$$f(x+y) = f(x) \cdot f(y)$$

Ex: 4) If $f: (R^+, \cdot) \rightarrow (R, +)$ defined by $f(x) = \log_e x$ $\forall x \in R^+$, Verify f is homomorphism or not.

$\Rightarrow$ $f(x) = \log_e x$ $f(y) = \log_e y$

$$f(x \cdot y) = \log_e xy \cdot \cancel{\log_e y}$$

$$= \log_e x + \log_e y$$

$$f(x \cdot y) = f(x) + f(y).$$

Theorem 1 : Let G and G' be two groups and $f: G \rightarrow G'$ homomorphism, then.

If e is identity of G then $f(e)$ is identity of G'
(f maps identity of G into identity of G')

$\Rightarrow$ Let $e \in G$ be identity of G.

$f(e.e) = f(e)$ and $f(e).f(e) = f(e.e)$
$= f(e)$

Since f is homomorphism, $f(e) \in G$.

$\therefore f(e.e) = f(e).f(e)$

If $e' \in G'$ is identity element of G'.

then $f(e) \in G'$, $f(e).f(e) = f(e)$.

$\therefore f(e)f(e) = e'f(e)$

$f(e) = e'$.

thus $f(e)$ is identity of e'.

Theorem 2: If G and G' are two groups of $f: G \to G'$ is a homomorphism, then If a^{-1} is the inverse of a in G, then $f(a^{-1}) = [f(a)]^{-1}$ $\forall\, a \in G$. (f preserves inverses).

Proof: For $a \in G$ $\exists\, a^{-1} \in G$ such that $a.a^{-1} = e$.

$\therefore f(a.a^{-1}) = f(e)$ $\forall\, a \in G$.

But $f(a)f(a^{-1}) = f(a.a^{-1})$ as f is homomorphism

$f(a).f(a^{-1}) = f(e)$ where e is identity of G'.

$\therefore f(a^{-1}) = [f(a)]^{-1}$

which is inverse of $f(a)$ in G'.

Theorem 3: If G and G' are two groups and $f: G \to G'$ is a homomorphism.

then, if H is a subgroup of G. then $f(H)$ is subgroup of G'. (f sends subgroup into subgroups)

Proof : Now $f(H) = \{ f(x) \in G' / x \in H \}$ is Subset of G'.

Now $e \in H$ $\therefore f(e) \in f(H)$ and $f(H) \neq \phi$.

let $f(x), f(y) \in f(H)$, then $x, y \in H$.

$$f(x)\,[f(y)]^{-1} = f(x).f(y)^{-1}$$
$$= f(xy^{-1})$$

But $x, y \in H$ $xy^{-1} \in H$ as H is subgroup of G.

$\therefore$ $f(xy^{-1}) \in f(H)$

$\forall$ $f(x), f(y) \in f(H) \Rightarrow f(x)\,[f(y)]^{-1} \in f(H)$.

$\therefore f(H)$ is subgroup of G'.

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Theorem 4 : If ~~H~~ is a cyclic Subgroup of G) G and G' are two groups and $f : G \to G'$ is a homomorphism.

then, if H is a cyclic subgroup of G, then $f(H)$ is cyclic sub group of G' [f preserves cyclicity]

$\Rightarrow$ Let H be the cyclic Subgroup of G, generated by 'a'. then $H = (a) = \{ a^n \mid n \in Z \}$ and $f(H)$ is Subgroup of G' (By above theorem (3)).

let $x \in f(H)$ then $x = f(h)$ $\forall$ $h \in H$.

But $h \in H \Rightarrow h = a^m$ $\forall$ $m \in Z$.

$$x = f(h) = f(a^m) = f(\underbrace{a.a.a \dots a}_{(m\ \text{times})})$$

$$= (\underbrace{f(a).f(a)..f(a) \dots f(a).}_{m\ \text{times}})$$

$$= [f(a)]^m$$

$\forall$ $x \in f(H)$ $x = [f(a)]^m$ $\forall$ $m \in Z$.

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$\therefore f(H)$ is cyclic subgroup of G'. generated by $f(a)$.

Note :

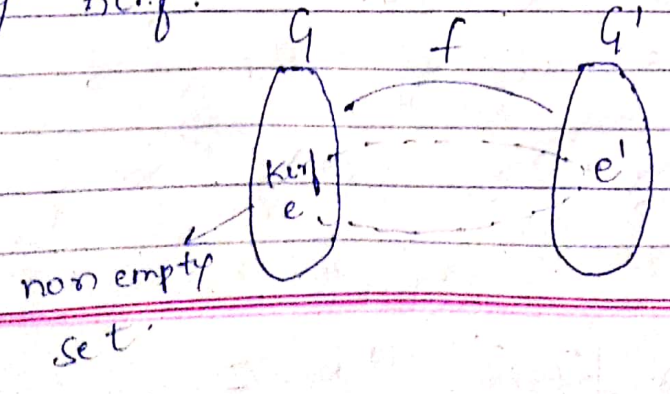
If G and G' are the two groups and $f: G \to G'$ is an isomorphism.

1) G and G' have same number of element.
2) If e is the identity of G, then $f(e)$ is identity of G'.
3) If H is a subgroup of G then $f(H)$ is subgroup of G'.
4) If H is normal subgroup of G, then $f(H)$ is also normal subgroup of G'.
5) The order of element a in G is same as order of $f(a)$ is G'.

Thus in this case G and G' can be treated as abstractly identical groups.

* Kernel of Homomorphism.

If G and G' are two groups and f (G to G') be a homomorphism from the group e' be the identity element of G'. the subset [K of G] $K(G)$ is defined by $K = \{a \in G \mid f(a) = e'\}$ is called Kernel of Homomorphism. and denoted by Ker f.



Examples:

1) If $(Z, +)$ i.e group of integers under addition Let $f: G \to G'$ defined by $f(x) = 2x$ find its Kernel.

$\Rightarrow$ $f: (Z, +) \to (2Z, +)$

$$Ker f = \{x \in G, f(x) = e'\}$$

$$= \{x \in (Z, +), 2x = 0\}$$

$$= \{x \in (Z, +), x = 0\}$$

$$K = \{0\}$$

2) Let $f: (R, +) \to (R^+, \cdot)$, $f(x) = e^x$. Find kernal.

$\Rightarrow$ $Ker f = \{x \in G, f(x) = e'\}$

$= \{x \in R, e^x = 1\}$

$= \{x \in R, e^x = e^0\}$

$= \{x \in R, x = 0\}$

$\therefore Ker f = \{0\}$

Theorem: If $f: G \to G'$ is a homomorphism from the group G into G' with Kernel K then: Prove that K is normal subgroup of G.

$\Rightarrow$ We shall prove this theorem by two steps

1) K is subgroup of G.

2) K is normal in G.

3) By defination of $ker f$, K is subgroup of G, $\therefore f(e) = e'$ where e and e' are identities of G and G'.

Hence $e \in K$ Thus $K \neq \phi$.

Now let $a, b \in K$.

then $f(a) = e'$ and $f(b) = e'$

$f(a.b^{-1}) = f(a).f(b^{-1})$ ($\because f$ is homomorphism)

$= e'.(e')^{-1}$

$= e'e' = e'$

Hence $ab^{-1} \in K$ and K is subgroup of G.

(2) To show that K is normal in G we shall show that $gkg^{-1} \in K$ and $\forall\, g \in G$ and $k \in K$.

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1})$$
$$= f(g).e'.f(g^{-1})$$
$$= f(g).f(g^{-1})$$
$$= f(g.g^{-1})$$
$$= f(e) = e'$$

$gkg^{-1} \in K \quad \forall\, g \in G$ and $k \in K$.

Theorem 2: If $f : G \to G'$ is a homomorphism from the group G in to G' with kernel K, then f is one-one iff $K = \{e\}$ where 'e' is identity element of G.

$\Rightarrow$ Let $f : G \to G'$ be one-one function

$$f(x_1) = f(x_2)$$
$$x_1 = x_2$$

Let $a \in K \Rightarrow f(a) = e'$ where e' is identity element of G.

$\therefore f(a) = f(e)$

$a = e$. ($\because f(e) = e'$)
(f is one-one)

$\therefore K = \{e\}$

Conversely, let $K = \{e\}$ we shall S.T f is one-one

Let $f(a) = f(b)$

$$f(a) \cdot [f(b)]^{-1} = f(b) \cdot [f(b)]^{-1}$$

$$f(a \cdot b^{-1}) = e'$$

$$a \cdot b^{-1} = e$$

$$(a \cdot b^{-1}) b = e \cdot b$$

$$a (b^{-1} \cdot b) = b.$$

$$a = b.$$

Hence f is one-one.

* Fundamental theorem of homomorphism :

"If f is a homomorphism of a group G in to group G' with kernel K then $f(G)$ is isomorphic to a factor (or Quotient) group G/K."

$\Rightarrow$ Since $f : G \to G'$ is a homomorphism set of $f(G)$ is a Subgroup of G'. Hence group by itself.

We shall define the map.

$$\phi : G/K \to f(G).$$

$$\phi(Ka) = f(a) \quad \forall \; a \in G \;, \; Ka \in G/K.$$

1) ϕ is well-defined.

Let $b \in Ka$ we shall show that $f(a) = f(b)$.

$b \in Ka \Rightarrow b = k_1 a \quad \forall \; k \in K.$

$\Rightarrow b = k_1 a.$

$\Rightarrow ba^{-1} = k_1$

$\Rightarrow f(k_1) = e' \quad (\because k \in \ker f = K).$

$\Rightarrow f(ba^{-1}) = e'$

$\Rightarrow f(b) \cdot f(a^{-1}) = e' \quad \Rightarrow f(b) \cdot [f(a)]^{-1} = e'$

$\Rightarrow f(b) = [f(a)] \cdot e'$

$f(b) = f(a).$

Hence ϕ is well defined.

(2) ϕ is one to one.

Let $\phi(Ka) = \phi(Kb)$

$$f(b) = f(a)$$

$$f(a)(f(b))^{-1} = e'$$

$$f(a \cdot b^{-1}) = e'$$

$$ab^{-1} = K$$

$$a \in K \cdot b$$

$$Ka = Kb.$$

Hence ϕ is one - one.

(3) ϕ is onto.

Let $b \in f(G)$ be arbitary.

$\therefore \exists\ a \in G$ such that

$$f(a) = \cancel{\phi}(b).$$

$\exists\ Ka \in G/K.$

Such that, $\phi(Ka) = f(a) = b.$

Hence ϕ is onto function.

(4) ϕ is homomorphism,

Consider, $\phi(Ka \cdot Kb) = \phi(Kab)$

$$= f(ab).$$

$$= f(a) \cdot f(b), \quad (\because f \text{ is homomorp}$$

$$= \phi(Ka) \cdot \phi(Kb).$$

Hence ϕ is homomorphism.

Hence ϕ is Isomorphism

and $f(G) \cong G/K. \Rightarrow$ Quotient group

Results On Isomorphism:

1) If $f : G \to G'$ be a Isomorphism of a Grou

G onto Group G' and 'a' is any element of G then the order of $f(a)$ equal to order of a.

$[o f(a) = o(a)]$

$\Rightarrow$ Let e and e' be the identity elements of G and G'

Case 1:

Let 'a' be a finite order say 'n'.
'n' is a least positive integer.
Such that $a^n = e$

Now $a^n = e$

$f(a^n) = f(e)$

$(f(a))^n = e'$ [$\because f$ is homomorphism]

$\Rightarrow f(a)$ is also finite order.

Let $o[f(a)] = m$, i.e m is least (+ve) integer

$(f(a))^m = e'$ $\therefore m$ divides n.

$(f(a))^m = e'$

$f(a^m) = f(e)$

$a^m = e$ {f is one-one}

But $O(a) = n$.

Thus 'n' divides 'm'.

$\therefore m$ divides n and n divides 'm'

Hence $m = n$.

$o(f(a)) = o(a)$.

Case ii)

Let 'a' be an infinite order.
and let $f(a)$ be a finite order say m.

That m is least (+ve) integer.

$$(f(a))^m = e'$$
$$f(a^m) = e'$$
$$f(a^m) = f(e)$$
$$a^m = e.$$

$\therefore$ a is of finite order.

$\therefore O[f(a)]$ is infinite

2) Let $f: G \to G'$ be an isomorphism. If G is abelian then G' is also abelian.

$\Rightarrow$ Let $a' b'$ and G' be arbitrary.

Since f is onto, $\exists$ $a, b \in G$

consider, $f(a) = a'$ $f(b) = b'$.

$$a'.b' = f(a).f(b)$$
$$= f(ab) = f(ba) \quad (\because G \text{ is abelian})$$
$$= f(b).f(a)$$

$$a'.b' = b'.a' \quad \forall\ a', b' \in G.$$

$\therefore$ G' is abelian.

3) Any infinite cyclic group is isomorphic to the group Z of integers under the addition.

$\Rightarrow$ Let G be infinite cyclic group having generator 'a'.

$$G = \{a^n / n \in Z\}$$

Defines, $f: G \to Z$

$\because f(a^n) = n \quad \forall\ a^n \in Z.$

i) f is homorphism.

Consider, $f(a^m \cdot a^n) = f(a^{m+n})$

$= m+n$

$f(a^m \cdot a^n) = f(a^m) + f(a^n)$

$f(a^m \cdot a^n) = f(a^m) + f(a^n)$

Hence f is homomorphism.

ii) f is onto

$\forall n \in Z$, $a^n \in G$, G being infinite group

$\forall n \in Z$, $\exists a^n \in G$ such that,

$f(a^n) = n$.

$\therefore f$ is onto.

iii) f is one-one.

$f(a^n) = f(a^m)$

$n = m$

$a^n = a^m$

$\therefore f$ is one-one.

Hence f is isomorphism.

UNIT-II

FOURIER SERIES

Syllabus:

Unit – II

Periodic functions, Fourier series of functions of period 2π and $2l$. Fourier series off odd and even functions, Half range sine and cosine series.

-10HRS

Fourier Series.

A function $f(x)$ is said to be an even fun if $f(-x) = f(x)$.

ex: $x^2, x^4, \ldots \cos x, \cos mx.\cos nx, \sin mx.\sin nx$
$2x^4, 3\sec x, \ldots$ etc are all even function.

A function $f(x)$ is said to be an odd function. $f(-x) = -f(x)$

$x^3, x^5, \ldots \sin x, \cos mx.\sin nx, 2x^3 - 3x + 4\sin x$ etc are all odd fun.

Periodic Function.

A function $f(x)$ is said to be periodic if it is defined for all real nos x's there is a +ve integer T, such that

$$f(x+T) = f(x)$$

The member T is called period of the function $f(x)$,

$\sin x, \cos x$ are periodic functions of period 2π.

k (const),

$\sin(x+2\pi) = \sin x, \quad \cos(x+2\pi) = \cos x$

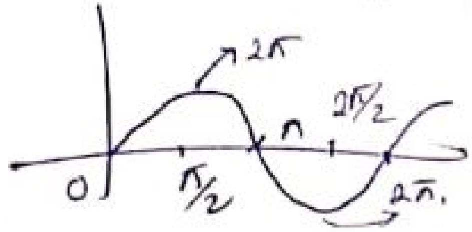
Also $f(x) = k, \quad f(x+2\pi) = k$

ex: $\sin(2\pi+x) = \sin(4\pi+x) = ---- = \sin x$

$\cos(2\pi+x) = \cos(4\pi+x) = ---- = \cos x$

$\sin x$ & $\cos x$ are periodic function of 2π

properties.



1. $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even fu^n

$= 0$ fu^n $f(x)$ is odd fu^n.

(1) $\int_{-\pi}^{\pi} \sin mx\,dx = 0 = \int_{-\pi}^{\pi} \cos mx\,dx = 0$

(2) $\int_{-\pi}^{\pi} \cos mx \cdot \sin nx\,dx = 0$

(3) $\int_{-\pi}^{\pi} \cos mx\,dx = 0 = 2\int_0^{\pi} \cos x\,dx = 0$

$\left[\because \int_{-\pi}^{\pi} \cos mx\,dx = 2\int_0^{\pi} \cos mx = 2\left[\frac{\sin mx}{m}\right]_0^{\pi}\right.$

$= \frac{2}{m}[\sin m\pi - \sin 0] = \frac{2}{m}(0-0) = 0\Big)$

(4) $\int_{-\pi}^{\pi} \cos mx \cdot \cos nx\,dx = 0$ if $m \neq n$

$= \pi$ if $m = n$.

(5) $\int_0^{2\pi} \sin mx\,dx = \int_0^{2\pi} \cos mx\,dx = 0$.

Ie.

(6). $\int_0^{2\pi} \cos mx . \cos nx \, dx = 0$ if $m \neq n$
$= \pi$ if $m = n$.

$\because \int_{-\pi}^{\pi} \cos 3x . \cos 2x \, dx = \frac{1}{2} \int_{-\pi}^{\pi} 2 \cos 3x \cos 2x \, dx$
$2 \cos A \cos B$

$= \frac{1}{2} [\cos 5x + \cos x] dx$

$= \frac{1}{2} \cancel{(\ldots)} . \int_0^{\pi} \cos 5x \, dx$

$= \frac{1}{2} (\sin 5x)_0^{\pi} = 0.$

if $m = n$

$\int_{-\pi}^{\pi} \cos 2x . \cos 2x = \int_{-\pi}^{\pi} \cos^2 (2x) \, dx = \int_{-\pi}^{\pi} \frac{1 + \cos 4x}{2} \, dx$

$= \frac{1}{2} \, 2 \int_0^{\pi} (1 + \cos 4x) dx = \left[x + \frac{\sin 4x}{4} \right]_0^{\pi}$

$= (\pi + 0) - (0 + 0) = \pi$ //

(7) $\int_0^{2\pi} \sin mx . \sin nx \, dx = 0$ if $m \neq n$
$= \pi$ if $m = n$.

NOTE:- The integrals of $\sin mx$, $\cos mx$, $\cos mx . \cos nx$, $\sin mx . \sin nx$ betn the limits $-\pi$ to π or 0 to π are always zero.

(8) $\int_{-\pi}^{\pi} \cos mx . \sin nx \, dx = 0$

(9) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$

(10) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$ be.

(11) $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \text{---} \text{---}$

C:' $\int x^3\sin x\,dx = x^3(-\cos x) - (3x^2)(-\sin x) + 6x(\cos x)$ — — ;

NOTE : The integraly of $\sin^2 mx, \cos^2 mx$ with limit $-\pi$ to π or 0 to π is π.

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Let $f(x)$ be continous single valued function. Which can be expressed in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

within the required range of values of the variables.

OR

If Defined in the interval $-\pi \leq x \leq \pi$ & $f(x)$ can be expanded into sines & cosines as follows.

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

$$\cdots + (a_n \cos nx + b_n \sin nx) \quad \text{--- A}$$

$$= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, b_3, b_4 \ldots$ are all constant & it is assumed that the series on RHS is uniformly convergent so that term by term integration is possible also it is assumed that RHS series is convergent. Then the above expansion series is called Fourier series are (expansion) of $f(x)$

In the interval $(-\pi, \pi)$ & $a_0, a_1, a_2, ---$ & $b_1, b_2, ---$ are called of Fourier constant

Dirichlet's condition

A function $f(x)$ defined on the interval $[-\pi, \pi]$ can be expressed in the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where a_0, a_n & b_n are real constant, if the following conditions called Dirichlet condition for convergence are satisfied in the interval.

1. $f(x)$ & its integral are finite & single values
2. $f(x)$ has discontinuities finite in no
3. $f(x)$ has maxima & minima finite in no.

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Fourier Series of functions with period 2π

let $f(x)$ be a periodic function with period 2π & $f(x)$ be represented by Fourier series by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

in the interval $[-\pi, \pi]$

To determine the coeff of a_0, a_n & b_n we assume that the series on the right hand side is uniformly convergent & it can be integrated term by term in the given integral.

$\therefore$ Integrating both side of (1) w.r.t 'x' betn the limits $[-\pi, \pi]$ we get.

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos nx\,dx\right]$$

$$+\sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} b_n \sin nx\,dx\right]$$

Now $\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0$ $(\because$ By (1)$)$

$\Rightarrow \frac{a_0}{2}\int_{-\pi}^{\pi} dx = \frac{a_0}{2}(\pi+\pi) = \pi a_0$

$(\sin x)_{-\pi}^{\pi}$
$\sin\pi - \sin(\pi)$
$= 0 - 0 = 0$

$\therefore \int_{-\pi}^{\pi} f(x)dx = \pi a_0$

$\therefore \boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$ — (I) / $\cos 180° = -1$.

To Find a_k.
Multiply equ^n (1) by $\cos kx$ & integrate bet^n the limits $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\cos kx \, dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx \, dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos nx . \cos kx \, dx\right] + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} b_n \sin nx \cos kx \, dx\right]$$

We have. $\int_{-\pi}^{\pi}\cos kx \, dx = 0 \longrightarrow$ By (I)

$\int_{-\pi}^{\pi}\cos nx . \cos kx \, dx = 0$ if $n \neq k$ By (4)

$\int_{-\pi}^{\pi}\sin nx . \cos kx \, dx = 0$ By (8). II

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$\therefore$ We get $\int_{-\pi}^{\pi} f(x).\cos kx\,dx = a_k \int_{-\pi}^{\pi} \cos^2 kx\,dx$ $\because$ since $n=k$

$\int_{-\pi}^{\pi} f(x).\cos kx\,dx = a_k \cdot \pi.$

$$\therefore \boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx} \quad \text{—— (II)}$$

Find b_k.

Again multiplying both sides of (1) by $\sin kx$ & integrate betn the limits $-\pi$ to π

We get

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin kx\,dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos nx \cdot \sin kx\,dx\right] + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \sin nx \sin kx\,dx\right]$$

We have $\int_{-\pi}^{\pi} \sin kx\,dx = 0$ By (I)

$\int_{-\pi}^{\pi} \sin nx.\sin kx\,dx = 0$ if $n \neq k$ By (III)

$\int_{-\pi}^{\pi} \cos nx.\sin kx\,dx = 0.$ By (II)

b_0.

$\therefore$ We get $\int_{-\pi}^{\pi} f(x).\sin kx\,dx = b_k \int_{-\pi}^{\pi} \sin^2 kx\,dx = b_k \pi$

$= b_k . \pi.$

$$\therefore \boxed{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin kx\,dx}$$

Thus the fourier series ~~of~~ for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos kx\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

The values of a_0, a_k, b_k, --- $k = 1, 2,$ -- given by above formulae are known as Euler's Formulae.

Ex:- Obtain the Fourier series of $f(x) = x^2$
$-\pi < x < \pi$ & $f(x+2\pi) = f(x)$ PT.

Solⁿ:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{3\pi}2\pi^3 = \frac{2\pi^2}{3}$$

$$a_k = \int_{-\pi}^{\pi} f(x).\cos kx\,dx \quad k = 1,2,3. \text{---}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2.\cos kx = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2}\,\underset{v}{\cos kx}\,dx$$

$$= \frac{2}{\pi}\left[x^2.\left(\frac{\sin kx}{k}\right) - 2x\left(\frac{-\cos kx}{k^2}\right) + 2\left(\frac{-\sin kx}{k^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{2\pi\cos\pi k}{k^2}\right) = \frac{4}{k^2}\cos k\pi, = \frac{4}{k^2}(-1)^n$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin kx\,dx \quad k = 1,2,3 \text{---}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin kx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\sin kx\,dx$$

$= 0$ ($\because x^2$ is even, $\sin kx$ is odd $\therefore x^2\sin kx$ is odd fun)

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi . \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \cdots \right] \quad (1)$$

$$\text{or } \frac{\pi^2}{3} + \sum_{1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

put $x = 0$. (2)

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots \qquad f(0) = 0^2 = 0.$$

$\because \cos 0 = 1$

put $x = \pi$ in (1)

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi .$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$\because f(\pi) = \pi^2, \quad \cos n\pi = (-1)^n$

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = 4 \sum_{1}^{\infty} \frac{1}{n^2} \qquad \because (-1)^{2n} = +1$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad (3)$$

add (2) & (3)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = 2.\frac{1}{1^2} + 2.\frac{1}{3^2} + 2.\frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$\mathcal{S}_l$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$$

if $n=1$ $\quad a_1 = \frac{4}{1}(-1) = -4$

$n=2$ $\quad a_2 = \frac{4}{4}(-1)^2 = \frac{4}{2^2} = 1$

$n=3$ $\quad a_3 = \frac{4}{9}(-1)^3 = -\frac{4}{3^2}$

$$\therefore x^2 = \frac{\pi^2}{3} + \left[-4\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \cdots\right]$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right] \quad (1)$$

put $x=0$

$$0 = \frac{\pi^2}{3} + 4\left[-1 + \frac{1}{2^2}(1) - \frac{1}{3^2}(1) + \cdots\right]$$

$$\frac{\pi^2}{3} = -4\left[-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{\pi^2}{4\times 3} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad (2)$$

put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{-(-1) + \frac{1}{2^2} - \frac{1}{3^2}(-1) + \cdots\right\}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right\}$$

$$\frac{2\pi^2}{3\times 4} = \left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right\}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \quad \text{--- (3)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

add (2) & (3)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = 2\cdot\frac{1}{1^2} + 2\cdot\frac{1}{3^2} + 2\cdot\frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{4} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

(4) Find e^x in $-\pi < x < \pi$ & hence prove (4)

$$\frac{\pi}{2}\operatorname{cosech}\pi = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

Solⁿ:- let $e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ — (A)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}[e^x]_{-\pi}^{\pi} = \frac{1}{\pi}[e^{\pi} - e^{-\pi}]$$

= Multiply & divide 2 = $\frac{2}{\pi}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right] = \frac{2}{\pi}\sinh\pi$.

$$\therefore \boxed{a_0 = \frac{2}{\pi}\sinh\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx$$

$$\left[\int e^{ax}\cos bx \, dx = \frac{1}{a^2+b^2} e^{ax}\{a\cos bx + b\sin bx\}\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx$$

$a = 1 \quad b = n$

$$= \frac{1}{\pi}\left[\frac{1}{1+n^2} e^x \cdot \{1 \cdot \cos nx + n \sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}\{\cos n\pi + n\underbrace{\sin n\pi}_{0}\} - e^{-\pi}\{\cos n\pi - n\sin n\pi\}\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}\{\cos n\pi\} - e^{-\pi}\{\cos n\pi\}\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[\cos n\pi \, (e^{\pi} - e^{-\pi})\right]$$

$$= \frac{\cos n\pi \times 2\sinh\pi}{\pi(1+n^2)}$$

$\therefore \sinh\pi = \frac{e^{\pi} - e^{-\pi}}{2}$

$\therefore e^{\pi} - e^{-\pi} = 2\sinh\pi$

$\Rightarrow a_n = \dfrac{2\sinh\pi(-1)^n}{\pi(1+n^2)}$ $\cos n\pi = -1$ if n is odd, 1 if n is even $\Rightarrow \frac{1}{\pi}\sinh\pi$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$ $= \frac{2\sinh\pi}{\pi}$

$= \frac{1}{\pi}\left[\frac{1}{1+n^2} e^x \{1.\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$ $= \frac{2\sinh\pi}{\pi}\Big[$

$= \frac{1}{\pi}\left[\frac{1}{1+n^2}\{e^{\pi}(\cancel{\sin n\pi} - n\cos n\pi) - e^{-\pi}(\cancel{-\sin n\pi} - n\cos n\pi)\right]$

$\because \sin n\pi = 0$ pu

$= \frac{1}{\pi(1+n^2)}\left[-n\cos n\pi(e^{\pi} - e^{-\pi})\right]$ $e^0 = \frac{2\sinh\pi}{\pi}$

$= \dfrac{-n(-1)^n 2\sinh\pi}{\pi(1+n^2)}$ or $\dfrac{-n(-1)^n \times 2\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)}{\pi(1+n^2)}$ $\because \sinh\pi = e^{\pi} - e^{-\pi}/2.$ $\frac{\pi}{2\sinh\pi} =$

$b_n = \dfrac{-2n(-1)^n.\sinh\pi}{\pi(1+n^2)}$

Substituting (A) we have

$e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$= \frac{1}{2} \times \frac{2}{\pi}\sinh\pi + \sum_{n=1}^{\infty}\left\{\dfrac{2\sinh\pi(-1)^n}{\pi(1+n^2)}\cos nx - \dfrac{2n(-1)^n\sinh\pi\sin nx}{\pi(1+n^2)}\right\}$

$= \frac{1}{\pi}\sinh\pi + \sum_{n=1}^{\infty}\frac{2\sinh\pi}{(1+n^2)\pi}\{(-1)^n\cos nx - n\times(-1)^n\sin nx\}$

$$\Rightarrow \frac{1}{\pi}\sinh\pi + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\{\cos nx - n\sin nx\}$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\{\cos nx - n\sin nx\}\right]$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}(\cos x - \sin x) + \frac{1}{5}(\cos 2x - 2\sin 2x) - \frac{1}{10}(\cos 3x - 3\sin 3x) - - - \right]$$

Put $x = 0$

$$e^0 = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}(1) + \frac{1}{5}(1-0) - \frac{1}{10}(1-0) - - \right]$$

$$\frac{\pi}{2\sinh\pi} = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\frac{\ldots nh\pi \sin nx}{\ldots 1+n^2)}\}$$

(5) Obtain the Fourier Series of

$f(x) = e^{-ax} \quad -\pi < x < \pi$.

where $f(x)$ is periodic with period 2π.

Solu^n:- The Fourier coeff are given by

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx$

$= \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{e^{-a\pi}}{-a} - \frac{e^{+a\pi}}{-a}\right]$

$= \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right] = \frac{2\sinh a\pi}{a\pi}$ $\quad \because \sinh a\pi = \frac{e^{a\pi} - e^{-a\pi}}{2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx$

$= \frac{1}{\pi}\left[e^{-ax}\left\{\frac{-a\cos nx + n\sin nx}{a^2+n^2}\right\}\right]_{-\pi}^{\pi}$

$\because \int e^{ax}\cos bx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$= \frac{1}{\pi}\left[\frac{e^{-a\pi}(-a\cos n\pi)}{a^2+n^2} - \frac{e^{a\pi}(-a\cos n\pi)}{a^2+n^2}\right]$

$= \frac{a\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$ $\quad \because \cos n\pi = (-1)^n$

$= \frac{2a\cos n\pi}{\pi(a^2+n^2)}\sinh a\pi = \frac{2a\sinh a\pi}{\pi(a^2+n^2)}(-1)^n$

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$$b_n = \frac{2a \sinh a\pi}{\pi(a^2+n^2)} (-1)^n$$

$$\therefore f(x) = e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$e^{-ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a \sinh a\pi}{\pi(a^2+n^2)} (-1)^n \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{2a \sinh a\pi}{\pi(a^2+n^2)} (-1)^n \sin nx.$$

put $x=0$ & $a=1$

$$1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$1 = \frac{\sinh \pi}{\pi} \left[1 + 2\sum_{1}^{\infty} \frac{(-1)^n}{n^2+1}\right]$$

$$\frac{\pi}{\sinh \pi} = 1 + 2\left[\frac{-1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \cdots\right]$$

$$\pi/\sinh \pi = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \cdots\right]$$

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H.W

(1) Find the Fourier expansion of $x + x^2$ in $-\pi < x < \pi$ & hence prove

$$\frac{\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

(2) If $f(x) = 0$ when $-\pi \le x \le 0$

$\sin x$ when $0 < x \le \pi$

S.T $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots = \frac{\pi - 2}{4}$

(3) $f(x) = \pi + x \quad -\pi < x < 0$

$\pi - x \quad 0 < x < \pi.$

(6) Find the Fourier Series expansion of $x+x^2$ in $-\pi < x < \pi$ & hence prove

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + - -$$

Solu^n:- Let $x + x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$f(-x) = -f(x)$ odd

$f(-x) = f(x)$ even

For odd $\int_{-\pi}^{\pi} x = 0$

even $\int_{-\pi}^{\pi} x^2 = 2\int_0^{\pi} x^2 dx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi}\Big[\underbrace{\int_{-\pi}^{\pi} x\,dx}_{\text{odd}} + \underbrace{\int_{-\pi}^{\pi} x^2\,dx}_{\text{even}}$$

$$= \frac{1}{\pi}\Big[0 + 2\int_0^{\pi} x^2 dx\Big]$$

$$= \frac{2}{\pi}\Big[\frac{x^3}{3}\Big]_0^{\pi} = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\Big[\underbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}_{\text{odd fun}} + \underbrace{\int_{-\pi}^{\pi} x^2 \cos nx\,dx}_{1 \times 1 \text{ even}}\Big)$$

$\cos(-\theta) = \cos\theta$

$$= \frac{1}{\pi}\Big[0 + 2\int_0^{\pi} x^2 \cos nx\,dx\Big)$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$\Rightarrow \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

put $x = \pi$,

$$= \frac{2}{\pi}\left[\left\{\pi^2 \frac{\sin n\pi}{n} - 2\pi\left(\frac{-\cos n\pi}{n^2}\right) + 2\left(\frac{-\sin n\pi}{n^3}\right)\right\} - \{0+0+0\}\right]$$

$\sin n\pi = 0.$

$$= \frac{2}{\pi} \times \frac{2\pi \cos n\pi}{n^2}$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}$$

$$\therefore \boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\,dx + \int_{-\pi}^{\pi} x^2 \sin nx\,dx\right]$$

even   odd.

$\sin(-\theta) = -\sin\theta.$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx\,dx + 0\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,d$$

$\because -x\sin(nx - x)$

$-x \times -\sin nx$

$+\sin nx$

$x^2 \sin(-nx)$

$-x^2\sin($

$$\frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n}+\frac{\sin n\pi}{n^2}\right]-(0-0)\right\}$$

$$=-\frac{2}{\cancel{\pi}}\times\frac{\cancel{\pi}\cos n\pi}{n}$$

$$b_n=\frac{-2(-1)^n}{n}$$

$$x+x^2=\frac{1}{2}\frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{n^2}\cos nx+\frac{(-4)(-1)^n}{2n}\sin nx\right\}$$

$$x+x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}4(-1)^n\left[\frac{\cos nx}{n^2}-\frac{\sin nx}{2n}\right]$$

This is true for $-\pi<x<\pi$

if $x=\pm\pi$ then

$$RHS=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left[(-1)^n\frac{\cos n(\pm\pi)}{n^2}-(-1)^n\frac{\sin n(\pm\pi)}{2n}\right]$$

$$=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left\{\frac{(-1)^n(-1)^n-0}{n^2}\right\}$$

$$=\pi^2/3+4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$=\pi^2/3+4\sum^{\infty}1/n^2=RHS.$$

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$RHS = \pi^2/3 + 4\sum_{n=1}^{\infty} 1/n^2$

But LHS is not defined as f(x) is not defined at $x=\pi$

when $x = \pm\pi$ f(x) is not defined

Now their is a ~~Rol~~ Role fourier series namely when $x = \pm\pi$ as follow the function can be defined

$$f(\pm\pi) = \frac{1}{2}\left[\lim_{x \to -\pi+0} f(x) + \lim_{x \to \pi+0} f(x)\right]$$

$$f(\pm\pi) = \frac{1}{2}\left[\lim_{x \to -\pi}(x+x^2) + \lim_{x \to \pi}(x+x^2)\right]$$

$$= \frac{1}{2}\left[-\pi+\pi^2 + \pi+\pi^2\right] = \frac{1}{\cancel{2}}\cancel{2}\pi^2 = \pi^2$$

$$\therefore \pi^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} 1/n^2$$

$$\pi^2 - \pi^2/3 = 4\sum_{n=1}^{\infty} 1/n^2$$

$$2\pi^2/3 = 4\sum_{n=1}^{\infty} 1/n^2$$

$$\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$$

$$\pi^2/6 = 1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \text{———}$$

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Up.

- Find Fourier series for the fun.

$$f(x) = \begin{cases} \pi + x & \text{if } -\pi < x < 0 \\ \pi - x & 0 \le x < \pi . \end{cases}$$

Solu^n:- The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(0+0) - \left(-\pi^2 + \frac{\pi^2}{2}\right) + \left(\pi^2 - \frac{\pi^2}{2}\right) - (0-0)\right]$$

$$= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[2\pi^2 - \pi^2\right]$$

$$a_0 = \frac{\pi^2}{\pi} = \pi$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\cos nx\,dx + \int_{0}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(\pi+x)\left(\frac{\sin nx}{n}\right) + (1)\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+\left[(\pi-x)\frac{\sin nx}{n} - (-1)\left(\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(0+\frac{1}{n^2}\right) - \left(0+\frac{\cos n\pi}{n^2}\right) + \left(0 - \frac{\cos n\pi}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right\}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^2} - \frac{2}{n^2}\cos n\pi\right] = \frac{2}{n^2\pi}\left(1-\cos n\pi\right)$$

$$\boxed{a_n = \frac{2}{n^2\pi}\left[1-(-1)^n\right]}$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \int_{0}^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(\pi+x)\left(\frac{-\cos nx}{n}\right)+\frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-\frac{\sin nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\pi\left(\frac{-1}{n}\right)+0\right\}-(0-0)+(0-0)-\left\{\pi\left(\frac{-1}{n}\right)-0\right\}\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\pi}{n}}+\cancel{\frac{\pi}{n}}\right]$$

$$\boxed{b_n = 0}$$

$\therefore$ From (1)

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left\{1-(-1)^n\cos nx + 0\right\}$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left\{1-(-1)^n\right\}\cos nx.$$

Q1.

Obtain the Fourier Series of

$$f(x) = \begin{cases} 1 & -\pi/2 < x < \pi/2 \\ -1 & \pi/2 < x < 3\pi/2 \end{cases} \quad \& \; f(x+2\pi) = f(x),$$

Soln:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

$$a_0 = \frac{1}{\pi} \int_{-\pi/2}^{3\pi/2} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi/2}^{\pi/2} 1\, dx + \int_{\pi/2}^{3\pi/2} (-1)\, dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(\frac{3\pi}{2} - \frac{\pi}{2}\right) \right] = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{3\pi/2} f(x) \cos nx \,.\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi/2}^{\pi/2} 1 \cdot \cos nx\, dx + \int_{\pi/2}^{3\pi/2} (-1) \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[\left\{ \frac{\sin nx}{n} \right\}_{-\pi/2}^{\pi/2} - \left\{ \frac{\sin nx}{n} \right\}_{\pi/2}^{3\pi/2} \right]$$

$$= \frac{1}{n\pi} \left[3 \sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right]$$

$\sin 90° = 1$

?-

$$a_1 = \frac{1}{\pi}\left[3\sin\frac{\pi}{2} - \sin\frac{3\pi}{2}\right] = \frac{4}{\pi}$$

$$a_2 = \frac{1}{\pi}\left[3\sin\pi - \sin 3\pi\right] = 0$$

$$a_3 = -\frac{4}{3\pi},\quad a_4 = 0,\quad a_5 = \frac{4}{5\pi} \text{ etc}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi/2}^{3\pi/2} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} 1.\sin nx\,dx + \int_{\pi/2}^{3\pi/2} (-1)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos nx}{n}\right\}_{-\pi/2}^{\pi/2} - \left\{\frac{\cos nx}{n}\right\}_{\pi/2}^{3\pi/2}\right]$$

$$= -\frac{1}{n\pi}\left[\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right]$$

$$= -\frac{1}{n\pi}\left[-2\sin 2n\pi \cdot \sin n\pi\right] = 0 \quad \forall \text{ all } n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{4}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - - - -\right\}$$

31.

③ ~~Obtain Fourier Series in the interval $(-\pi,\pi)$ for all fun $f(x)=x$ Hence P.T $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+---$~~

Solun:-

H.W.

(2) Obtain Fourier Series for the fun

$$f(x) = \begin{cases} -k & -\pi<x<0 \\ k & 0<x<\pi \end{cases} \quad \& \ f(x+2\pi)=f(x)$$

Solun:- Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}k\,dx+\int_{0}^{\pi}k\,dx\right.$

$=\frac{1}{\pi}\left[-k(x)_{-\pi}^{0}+k(x)_{0}^{\pi}\right]=0.$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x).\cos nx\,dx$

$=\frac{1}{\pi}\int_{-\pi}^{0}-k.\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}k.\cos nx\,dx.$

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$= \frac{1}{\pi}\left[-k\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + k\left\{\frac{\sin nx}{n}\right\}_{0}^{\pi}\right] = 0.$

$(\sin n\pi = 0 = \sin 0)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} -k.\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} k.\sin nx\,dx$

$= \frac{1}{\pi}\left[-k\left\{\frac{-\cos nx}{n}\right\}_{-\pi}^{0} + k\left\{\frac{-\cos nx}{n}\right\}_{0}^{\pi}\right]$

$= \frac{1}{\pi}\left[\frac{k}{n}\{1-\cos n\pi\} - \frac{k}{n}\{\cos n\pi - 1\}\right]$

$= \frac{k}{n\pi}\left[1-\cos n\pi - \cos n\pi + 1\right]$

$= \frac{2k}{n\pi}\left[1-\cos n\pi\right]$

$\cos n\pi = (-1)^n$, $\therefore b_k = \frac{2k}{n\pi}\left[1-(-1)^n\right]$

$b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$, $b_4 = 0$, $b_5 = \frac{4k}{5\pi}$ ---

$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + --\right]$

Fourier Series of functions with period $2L$.

Let $f(x)$ be a function with period $2L$.

$f(x+2L) = f(x)$ $\forall$ all x

put $x = \frac{Lt}{\pi}$

$$f\left(\frac{Lt}{\pi} + 2L\right) = f\left(\frac{Lt}{\pi}\right)$$

$$f\left(\frac{Lt + 2L\pi}{\pi}\right) = f\left(\frac{Lt}{\pi}\right)$$

$$f\left[\frac{L}{\pi}(t + 2\pi)\right] = f\left(\frac{Lt}{\pi}\right)$$

$\therefore f\left(\frac{L}{\pi}t\right)$ is a function with period 2π,

Hence $f\left(\frac{L}{\pi}t\right)$ may be expanded in a Fourier series in the interval $-\pi \le t \le \pi$ in the form.

$$f\left(\frac{L}{\pi}t\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) dt$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) \cos nt \, dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) \sin nt \, dt$$

Since $x = \frac{Lt}{\pi}$ we have $-\pi \leq t \leq \pi$,

$-\frac{L}{\pi} \cdot \pi \leq \frac{L}{\pi} \cdot t \leq \frac{L}{\pi} \cdot \pi$ i.e $-L \leq x \leq L$

Also $t = \frac{\pi x}{L} \Rightarrow dt = \frac{\pi}{L} dx$

Thus we have

$$a_0 = \frac{1}{\pi} \int_{-L}^{L} f(x) \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \cdot \frac{\pi}{L} dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\pi} \cdot \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \frac{\pi}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$f\left(\frac{Lt}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Fourier Series of fun with period 2L. (continue)

(i) Find the Fourier series for

$$f(x)=\begin{cases}-1 & -1<x<0\\ 2x & 0<x<1\end{cases} \quad \& \quad f(x+2)=f(x)$$

Solu^n:- For $f(x)$, period $=2L=2$

$L=1$

The Fourier series for $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{1}\right)+\sum_{n=1}^{\infty} b_n\sin(n\pi x)$$

$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\,dx=\int_{-1}^{1} f(x)\,dx$$

$$=\int_{-1}^{0}-1\,dx+\int_{0}^{1}2x\,dx=-[x]_{-1}^{0}+[x^2]_{0}^{1}=0$$

$$\boxed{a_0=0}$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x).\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$a_n=\int_{-1}^{1} f(x).\cos(n\pi x).dx$$

$$=\int_{-1}^{0}(-1).\cos(n\pi x)dx+\int_{0}^{1}2x\cos(n\pi x)\,dx.$$

$$= \left[\frac{-\sin n\pi x}{n\pi}\right]_{-1}^{0} + \left[2x.\frac{\sin n\pi x}{n\pi} - 2\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 0 + \frac{2\cos n\pi}{n^2\pi^2} - \frac{2}{n^2\pi^2}$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

If n is even $(-1)^n - 1 = 0 \quad \therefore a_n = 0$

If n is odd $(-1)^n - 1 = -1-1 = -2$

$$a_n = \frac{-4}{n^2\pi^2}, \quad (n = 1, 3, 5, 7, \ldots)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x).\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_{-1}^{1} f(x).\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0}(-1).\sin(n\pi x)\,dx + \int_0^1 2x\,\sin(n\pi x)\,dx$$

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[2x\left(-\frac{\cos n\pi x}{n\pi}\right) - 2\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{1}{n\pi}(1-\cos n\pi) - \frac{2}{n\pi}\cos n\pi.$$

$$= \frac{1}{n\pi}\left[1 - 3\cos n\pi\right] = \frac{1}{n\pi}\left[1 - 3(-1)^n\right]$$

$\therefore$

$a_1 = \frac{-4}{\pi^2}$, $a_3 = \frac{-4}{9\pi^2}$, $a_5 = \frac{-4}{25\pi^2}$, ---

$b_1 = 4/\pi$, $b_2 = -1/\pi$, $b_3 = 4/3\pi$, ---

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$= \frac{-4}{\pi^2}\left\{\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right\}$$

$$+ \frac{1}{\pi}\left\{4\sin \pi x - \sin 2\pi x + \frac{4}{3}\sin 3\pi x + \cdots\right\}$$

2) Find Fourier expansion for the fun $f(x) = x - x^2$, $-1 < 0 < 1$

Solu^n:- For $f(x)$ period $2L = 2$, $\therefore L = 1$

$\therefore$ The Fourier Series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

$$= \int_{-1}^{1} (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

period 2L & continue.

(3) Find the Fourier expansion for the fun

$$f(x) = \begin{cases} -1 & -3 < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < 3. \end{cases}$$

Solⁿ: For given period $2L = 6$

$L = 3$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

where $a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx$

$$a_0 = \frac{1}{3}\int_{-3}^{0} f(x)\,dx + \frac{1}{3}\int_{0}^{3} f(x)\,dx$$

$$= \frac{1}{3}\int_{-3}^{0} (-1)\,dx + \frac{1}{3}\int_{0}^{3} 1\,dx = 0$$

$$\therefore \boxed{a_0 = 0}$$

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cdot\cos\left(\frac{n\pi x}{3}\right)dx$$

$$a_n = \frac{1}{3}\int_{-3}^{0} (-1)\cdot\cos\left(\frac{n\pi x}{3}\right)dx + \frac{1}{3}\int_{0}^{3} \cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{1}{3}\left[\left(\frac{-\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}\right)_{-3}^{0} + \left[\frac{\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}\right]_{0}^{3}\right\}$$

$$= \frac{1}{3}\left[\left\{\left(-\frac{n\pi}{3} \times \sin\left(\frac{n\pi x}{3}\right)\right]_{-3}^{0} + \left[\frac{n\pi}{3}\cdot\sin\left(\frac{n\pi x}{3}\right)\right]_{0}^{3}\right.$$

$= 0$.

$\therefore \boxed{a_n = 0}$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\cdot \sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{1}{3}\int_{-3}^{0} (-1).\sin\left(\frac{n\pi x}{3}\right)dx + \frac{1}{3}\int_{0}^{3} \sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{1}{3}\left[\left[\frac{\cos\left(\frac{n\pi x}{3}\right)}{\frac{n\pi}{3}}\right]_{-3}^{0} + \left[\frac{-\cos\left(\frac{n\pi x}{3}\right)}{\frac{n\pi}{3}}\right]_{0}^{3}\right]$$

$$= \frac{1}{3}\left[\frac{3}{n\pi}\left[1-(-1)^n\right] - \frac{3}{n\pi}\left\{(-1)^n - 1\right\}\right]$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right] - \frac{1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$\therefore b_n = \frac{4}{n\pi}$ when n is odd

$b_n = 0$ when n is even

$\therefore$ Fourier series is

$$f(x) = \frac{4}{\pi}\left\{\sin\frac{\pi x}{3} + \frac{1}{3}\sin \pi x + \frac{1}{5}\sin\frac{5\pi x}{3} + \dots\right\}$$

Fourier Series of even & odd Function

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$.

ex: $\cos x, x^2, x^3 \sin x, x^2 \cos x$.

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$.

ex: $\sin x, x^3 \cos x, x^3, x^2 \sin x$.

The Geometric characteristic of the graph of even function $f(x)$ is that graph is symmetric w.r.t y axis & for the graph of an odd function we have symmetric w.r.t the origin.

propertie

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & f(x) \text{ is odd.} \end{cases}$$

L.

Q1.

Let $f(x)$ be a periodic function with period $2l$. We consider two cases.

Case I :- Let $f(x)$ be an Even function

The Fourier Series for is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$\therefore$ $f(x)$ is even.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx.$$

Again $f(x)$ is even & $\cos\left(\frac{n\pi x}{l}\right)$ is even

$\therefore f(x) \cdot \cos\left(\frac{n\pi x}{l}\right)$ is even

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

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&l

Now, $f(x)$ is even & $\sin\left(\frac{n\pi x}{l}\right)$ is odd.

$\therefore$ $f(x).\sin\left(\frac{n\pi x}{l}\right)$ is odd.

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin\left(\frac{n\pi x}{l}\right).dx$$

$b_n = 0.$

Hence for even periodic fu^n $f(x)$, with period $2l$, we have Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$\therefore$ $f(x)$ is even periodic fu^n with period 2π. We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$

case 2:- Let $f(x)$ be an odd function

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = 0$$

$f(x)$ is odd & $\cos\left(\frac{n\pi x}{l}\right)$ is even.

$\therefore f(x)\cos\left(\frac{n\pi x}{l}\right)$ is odd.

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$= 0$

Again $f(x)$ is odd & $\sin\left(\frac{n\pi x}{l}\right)$ is odd.

$\therefore f(x).\sin\left(\frac{n\pi x}{l}\right)$ is even.

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin\left(\frac{n\pi x}{l}\right).dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Hence for odd periodic $f(x)$, with period $2l$, we have Fourier series.

$$f(x) = \sum_{n=1} a_n \sin\left(\frac{n\pi x}{l}\right).$$

where $b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right).dx$.

$\therefore$

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Even & odd fun

$$= \begin{cases} \frac{-4l}{n^2\pi^2} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

$\therefore$

$$f(x) = |x| = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} + \frac{\cos\left(\frac{3\pi x}{l}\right)}{3^2} + \cdots\right]$$

$\therefore$ period $= 2\pi$,

$$f(x) = |x| = \pi/2 - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

2) Obtain the Fourier series for the expansion of $f(x) = x^2$, $-\pi \le x \le \pi$, $f(x+2\pi) = f(x)$

Solⁿ:- Here $f(x) = x^2$ is even funⁿ. Hence. We have Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

put ($l = \pi$) given

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

Even & odd fun (3)

$$= \begin{cases} -\dfrac{4l}{n^2\pi^2} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

$\therefore$

$$f(x) = |x| = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} + \frac{\cos\left(\frac{3\pi x}{l}\right)}{3^2} + \cdots\right]$$

$\therefore$ period $= 2\pi$,

$$f(x) = |x| = \pi/2 - 4/\pi\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

(2) Obtain the Fourier series for the expansion of $f(x) = x^2$, $-\pi \le x \le \pi$, $f(x+2\pi) = f(x)$.

Solⁿ:- Here $f(x) = x^2$ is even funⁿ. Hence. We have Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

put ($l = \pi$) given

where

$$a_0 = 2/\pi \int_0^{\pi} f(x)\,dx = 2/\pi \int_0^{\pi} x^2\,dx$$

$$= 2/\pi \left[\frac{x^3}{3}\right]_0^{\pi} = 2\pi^2/3$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2.\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2.\frac{\sin nx}{n} - \left(\frac{2x}{n^2}\right)(-\cos nx) + \frac{2}{n^3}(-\sin nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] \qquad \because \sin n\pi = 0$$

$$= \frac{4}{n^2}\cos n\pi$$

$$= \begin{cases} -\frac{4}{n^2} & \text{when } n \text{ is odd} \\ \frac{4}{n^2} & \text{when } n \text{ is even.} \end{cases}$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right]$$

put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} - 4\left[\cos\pi - \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi - \cdots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = -4\left[-1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2} + \cdots\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

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(2) $f(x) = \begin{cases} \frac{\pi - x}{2} & 0 < x < \pi \\ -\left(\frac{\pi + x}{2}\right) & -\pi < x < 0 \\ 0 & x = 0 \text{ or } \pm\pi, \end{cases}$

$\Rightarrow$ Here $f(-x) = \begin{cases} \frac{\pi + x}{2} & 0 < x < \pi \\ -\left(\frac{\pi - x}{2}\right) & -\pi < x < 0 \\ 0 & x = 0 \text{ or } \pm\pi, \end{cases}$

$f(-x) = -f(x)$.

$\therefore f(x)$ is odd function

Hence the Fourier Series

$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ ($\because l = \pi$).

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot \sin(nx)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi - x}{2}\right)\sin(nx)\,dx$

$= \frac{2}{\pi}\left[\frac{\pi - x}{2}\left(-\frac{\cos nx}{n}\right) - \left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{2}{\cancel{\pi}} \times \left[\frac{\cancel{\pi}}{2n}\right] = \frac{1}{n}$

$$f(x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} \frac{\pi - x}{2} & 0 < x < \pi \\ -\left(\frac{\pi + x}{2}\right) & -\pi < x < 0 \\ 0 & x = 0 \text{ or } x = \pm\pi \end{cases}$$

Even & odd fun^{n}.

(2) Obtain the Fourier Series of $f(x) = |x|$ in $(-\pi, \pi)$.

Solu^{n}:- $f(x) = |x|$ $(-\pi, \pi)$

which is even fun^{n}.

$f(x) = |x| = x$ where $x \geq 0$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi^2 - 0\right] = \pi$$

Find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

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$$\therefore |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}\cos nx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\left\{1(-2)\cos x + 0 + \frac{1}{3^2}(-2)\cos 3x + \frac{1}{5^2}(-2)\cos 5x\right\}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x\right\}$$

put $x = \pi$.

$$\pi = \frac{\pi}{2} - \frac{4}{\pi}\left(-1 - \frac{1}{3^2} - \frac{1}{5^2} - \cdots\right)$$

$$\pi - \frac{\pi}{2} = \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

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③ Obtain Fourier Series for the fun $f(x) = x.\cos x$. $(-\pi, \pi)$

soluⁿ:- Given fun is $f(x) = x\cos x$ ~~is~~ is odd fun for,

$$f(-x) = -x\cos(-x) = -f(x)$$

$$\therefore a_0 = 0 \ \& \ a_n = 0 \quad n = 1, 2, 3 \ldots$$

The Fourier Series is

$$f(x) = \sum_{n=1}^{\infty} b_n . \sin nx \quad \text{———} \ (1)$$

Now find b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x.\cos x . \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\left[\frac{1}{2}\left[\sin(nx+x) + \sin(nx-x)\right]\right\} dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left\{\left[\sin(n+1)x + \sin(n-1)x\right]\right\} dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[x.\left(\frac{-\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2}\right) + \left[x.\left(\frac{-\cos(n-1)x}{n-1}\right) + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi(-1)^{n+1}}{n+1} + 0 + 0 - \frac{\pi(-1)^{n-1}}{n-1} + 0 + 0\right]$$

$$= \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}(-1)^{n+1}}{n+1} - \frac{\cancel{\pi}(-1)^{n-1}}{n-1}\right]$$

$$= \left[\frac{(-1)^{n+1}(-1)^{1}}{n+1} - \frac{(-1)^{n}(-1)}{n-1}\right] \rightarrow \text{check how } (-1)$$

$$= \left[\frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n}}{n-1}\right] \quad (+)\text{ become}$$

$$= (-1)^{n}\left[\frac{(-1)^{2}}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^{n}\left[\frac{n-1+n+1}{n^2-1}\right]$$

$$= (-1)^{n}\frac{2n}{n^2-1}$$

$\therefore$ Fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$x\cos x = \sum_{n=1}^{\infty} \frac{2n(-1)^{n}}{n^2-1} \times \sin nx$$

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$\mathcal{E}$p.

Examples:-

(1) Find Fourier series for the periodic function f(x) with period $2l$.

where $f(x) = |x|$, $-l \le x \le l$

Solu^n:- Here $f(x) = |x|$ is even fu^n.

Hence For even periodic fu^n f(x) with period $2l$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{l}\int_0^l x\,dx = l$$

$$a_n = \frac{2}{l}\int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x.\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{lx}{n\pi}\sin\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{lx}{n\pi}\sin(n\pi) + \frac{l^2}{n^2\pi^2}\cos(n\pi) - \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{2l}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

In particular $f(x)$ is an odd periodic fu^n with period 2π,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$,

$\therefore$ Formula.

(1) If $f(x)$ is even function.

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx.$$

$$a_n = \frac{2}{l}\int_0^{l} f(x).\cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).$$

(2) If $f(x)$ is odd function.

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

∂_1.

Examples:-

(1) Find Fourier series for the periodic function $f(x)$ with period $2l$

where $f(x) = |x|$, $-l \le x \le l$

Solu^n:- Here $f(x) = |x|$ is even fu^n,

Hence For even periodic fu^n $f(x)$ with period $2l$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{l}\int_0^l x\,dx = l$$

$$a_n = \frac{2}{l}\int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x.\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{lx}{n\pi}\sin\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{lx}{n\pi}\sin(n\pi) + \frac{l^2}{n^2\pi^2}\cos(n\pi) - \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{2l}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

B.P.

(1) Find a_0 in the Fourier Expansion of $f(x) = x + x^2$ in $(-\pi, \pi)$.

$\Rightarrow$ Given $f(x) = x + x^2$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

Find a_0.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^3}{\cancel{\pi}\, 3}$$

$$a_0 = \frac{2\pi^2}{3}$$

Half-range Sine & Cosine Series

The fun $f(x)$ in range $(0, \pi)$ in a Fourier series of period 2π.

OR In the range $(0, L)$ in Fourier series of period $2L$.

In a periodic fun with period $2l$ is defined only in the interval $(0, l)$ then it can be expanded to in a series containing only sine or cosine function. Such Fourier series is called a Fourier cosine half range or Fourier sine half range series.

Cosine expansion :- To find the expansion of $f(x)$, in the range $0 < x < L$ to contain only cosine terms. We extend the fun $f(x)$ by reflecting it in the interval

Half Range Sine Series

If we required to expand of f(x) as a Sine Series in $(0, l)$ then we extend the function reflecting in the origin $\therefore f(x) = -f(-x)$

Then the extended function is odd in $(-l, l)$ & the expansion will give the desired Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

NOTE :-

(1) It has to be written as Fourier Sine half range series in $(0, l)$ then $f(x)$ is treated as odd function in $(-l, l)$

(2) Suppose $f(x)$ is a function in the interval $(0, \pi)$ & $-f(-x)$ in interval $(-\pi, 0)$ then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Half-Range Cosine Series

If it is required to express $f(x)$ as a cosine series in $(0, l)$ we extend the function reflecting it in y-axis, so that

$$f(-x) = f(x)$$

Then the extended function is even in $(-l, l)$ & its expansion will give required Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l} \int_0^l f(x)\,dx$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

NOTE:- (1) It has to be written Fourier Cosine half-range series in $(0, l)$ then $f(x)$ is treated as even function $(-l, l)$

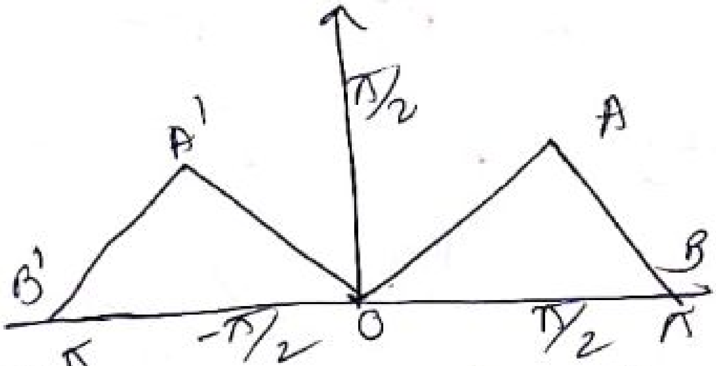
(2) $f(x)$ is a f^n in interval $(0, \pi)$ & $f(-x)$ in $(-\pi, 0)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx$$

1. Represent the following fun $f(x)$ as Fourier cosine half range series.

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$$

Solⁿ:- The graph of $f(x)$ in $0 < x < \pi$ is OAB in fig.



let us extend the fun $f(x)$ in the interval $(-\pi, 0)$ so that the new fun is symmetrical about y-axis (graphically shown by $B'A'O$) Thus $f(x)$ is an even fun in $[-\pi, \pi]$ & will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\,dx$$

$$= \pi/$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\cancel{\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right.$$

$$\left. - \frac{\pi}{2n}\cancel{\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$a_1 = 0$, $a_2 = -\frac{8}{4\pi}$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$,

$a_6 = \frac{8}{\pi 6^2}$ etc.

$$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{8^2}\cos 8x + \cdots\right]$$

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Q.

(1) Represent the function $f(x)$ as Fourier Sine half range series where $f(x) = x, \quad 0 < x < \pi/2$

$f(x) = \pi - x \quad \pi/2 < x < \pi.$

Solu^n:- We shall extend the fu^n $f(x)$ shown by $B'A'O$ in the fig. Such that extended function $f(x)$ represents an odd function. because the curve is symmetrical about the origin

Thus the Fourier expansion for $f(x)$ over the full period 2π will contain only sine terms & is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{nx}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[-(\pi-x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4}{n^2\pi} \times \sin\frac{n\pi}{2}$$

$b_2 = b_4 = b_6 = \dots = 0$

$b_1 = \frac{4}{1^2\pi}$, $b_3 = -\frac{4}{3^2\pi}$, $b_5 = \frac{4}{5^2\pi}$ ---- etc

Hence Require sine series $f(x)$ over half Range $[0, \pi)$

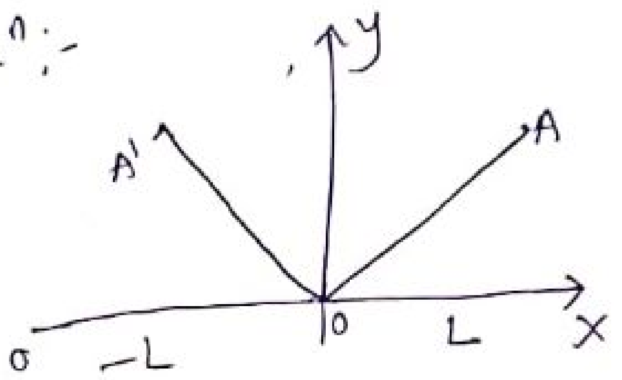
$$f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \dots\right]$$

8e

(3) Represent the following function $f(x)$ as a Fourier cosine series

$$f(x) = x, \quad 0 \le x \le L.$$

Soln:-



We shall extend the function $f(x)$ shown in the fig by OA' such that extended function $f(x)$ represents as even function because the curve is symmetrical about the y-axis

Thus the Fourier expansion for $f(x)$ over the full period $2L$ will contain only cosine terms & is given by

$\because \sin n\pi = 0, \quad \cos n\pi = (-1)^n$

$\therefore a_n = 0$ when n is even

when n is odd

$$a_n = \frac{2L}{n^2\pi^2}(-2) = -\frac{4L}{n^2\pi^2} \quad \therefore n = 1, 3, 5, \ldots$$

Hence Required Fourier Cosine series of $f(x)$ over the half range $[0, L]$ is

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\left\{\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right\}$$

5) 2018.

Find half-range cosine series for the function $f(x) = (x-1)^2$ in $(0,1)$. Hence P.T $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

Solu^n:- Fourier series is

$f(x) = (x-1)^2$ in $(0,1)$

For Fourier half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad (1)$$

where $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 (x-1)^2\,dx$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}\left\{(1-1)^3 - (0-1)^3\right\}$$

$a_0 = \frac{2}{3}$

$a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$

$$= 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1) - \left(\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]$$

$$= 2\left[(0-0+0) - (0) - 2\left(\frac{-1}{n^2\pi^2}\right) + 0\right]$$

$$= a_n = 2\left(\frac{2}{n^2\pi^2}\right) = \frac{4}{n^2\pi^2}$$

put value of a_0 & a_n in (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$(x-1)^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$$

$$= \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\pi x$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left\{\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \cdots\right\}$$

⑤ continue.

put $x=1$ then

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\left[-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \cdots\right] \quad (2)$$

put $x=0$

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right\} \quad (3)$$

Substract (3) − (2) givy

$$1 = 0 + \frac{4}{\pi^2}\left\{2 + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \cdots\right)$$

$$1 = 0 + \frac{8}{\pi^2}\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right\}$$

$$\pi^2 = 8\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right\}$$

2019

⑥ Find the half-range sine & cosine series for the function $f(x) = \pi - x$, $0 < x < \pi$

Pf:- Given function is $f(x) = \pi - x$, in $(0, \pi]$

(1) Half Range Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin nx \, dx$

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(0+0) - \left(\frac{-\pi}{n}\right)\right] = \frac{2}{n}$$

$$\therefore b_n = \frac{2}{n}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$(\pi - x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

(ii) Half Range cosine Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\, dx$

$$= \frac{2}{\pi}\left[\frac{(\pi-x)^2}{-2}\right]_0^{2} = \frac{2}{\pi}\left[0 + \frac{\pi^2}{2}\right]$$

$a_0 = \pi.$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \cos n\pi x\, dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 - \frac{\cos n\pi}{n^2}\right\} - \left\{0 - \frac{1}{n^2}\right\}\right]$$

$$= \frac{2}{n^2\pi}\left[1 - (-1)^n\right]$$

2M. 2017. B.E.

(4) Find the half Range cosine series of $f(x) = x$ in $(0, 2)$ (only find a_0) i.e. 2M.

Solun: Let $f(x) = x$ in $(0, 2)$

For Fourier half range cosine series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad \text{--- (1)}$$

where $a_0 = 2/2 \int_0^2 f(x)\,dx = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2$

$$a_0 = 2.$$

$$a_n = \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[x \frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2} - 1\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)\right]_0^2$$

$$= \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left[0 + \frac{4}{n^2\pi^2}\cos n\pi\right] - \left[0 + \frac{4}{n^2\pi^2}\cos 0\right]$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

put values of a_0 & a_n in (1) we get

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\{(-1)^n - 1\}\cos\left(\frac{n\pi x}{2}\right)$$

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\{(-1)^n - 1\}\cos\left(\frac{n\pi x}{2}\right)$$

$\therefore$ OR

$$x = 1 + \frac{4}{\pi^2}\left[\frac{-2\cos \pi x}{1^2} + 0 + \frac{-2\cos\left(\frac{3\pi x}{2}\right)}{3^2} + 0 + \frac{-2\cos\left(\frac{5\pi x}{2}\right)}{5^2} + \right.$$

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(7) Find half-Range sine & cosine series for the function $f(x) = 2x-1$ in $(0,\pi)$

Solu^n:- Given function is $f(x) = 2x-1$

(i) Half-Range Sine series

$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where $b_n = \frac{2}{1}\int_0^1 (2x-1)\sin n\pi x\,dx$

$= 2\left[(2x-1)\left(\frac{-\cos n\pi x}{n\pi}\right) - 2\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$

$= 2\left[\left\{\frac{-\cos n\pi - 0}{n\pi}\right\} - \left\{\frac{1}{n\pi} - 0\right\}\right]$

$= 2\left[\frac{-1}{n\pi}(\cos n\pi + 1)\right]$

$= \frac{-2}{n\pi}\{1+(-1)^n\}$

$= \sum_{n=1}^{\infty} b_n \sin n\pi x$

$(2x-1) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\{1+(-1)^n\}\sin n\pi x$

(ii) Half-Range cosine series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$

where $a_0 = \frac{2}{1}\int_0^1 (2x-1)\,dx$

$= 2\left[\frac{2x^2}{2} - x\right]_0^1 = 2[(1-1)-0]$

$a_0 = 0$

$a_n = \frac{2}{1}\int_0^1 (2x-1)\cos n\pi x\,dx$

$= 2\left[(2x-1)\left(\frac{\sin n\pi x}{n\pi}\right) - (2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$

$= 2\left[\left\{0 + \frac{2\cos n\pi x}{n^2\pi^2}\right\} - \left\{0 + \frac{2}{n^2\pi^2}\right\}\right]$

$= \frac{4}{n^2\pi^2}\{(-1)^n - 1\}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$

$(2x-1) = 0 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\{(-1)^n - 1\}\cos n\pi x$

$(2x-1) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\{(-1)^n - 1\}\cos n\pi x$

UNIT-III

Fourier Transforms

Syllabus:

Unit – III

Finite sine and Cosine transforms.

- 10HRS

FOURIER TRANSFORMS

In this topic we shall discuss the Fourier sine and cosine transforms & their properties. These Transforms are appropriate for problems over finite intervals in a variable in which the function or its derivative are prescribed on boundary.

Finite Fourier sine and cosine transforms are main concept to study in interval $(0, l)$ & $(0, \pi)$ which are Finite. Here we use the formula for Fourier sine & cosine series with finite range.

The Finite Fourier Sine Transforms of $f(x)$

It is useful for problems involving boundary conditions of heat distribution on two parallel boundaries.

Defination:- The finite Fourier sine transform of $f(x)$ in $(0, l)$ is defined by

$$F_s(n) = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

where n is positive integer.

The function $f(x)$ is then called the inverse finite Fourier sine transform of $F_s(n)$ & is given by

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{l}\right). \quad (1)$$

The above formula is obtained from Fourier Sine Series

i.e $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

NOTE:- The finite Fourier sine transform of $f(x)$ in $(0, \pi)$ is defined by

$$F_s(n) = \int_0^{\pi} f(x) \sin nx \, dx$$

where $f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{l}\right)$

& n is positive integer.

The Finite Fourier Cosine Transform of $f(x)$

The finite Fourier Cosine Transform is useful for problems in which the velocity normal to two parallel boundaries are among the boundary conditions

Defination:- The finite Fourier cosine transform of $f(x)$ in $(0, l)$ is defined by

$$F_c(n) = \int_0^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx.$$

where n is positive integer.

The function $f(x)$ is then called inverse finite Fourier cosine transformation of $F_c(n)$ & is given by

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{l}\right) \quad (1)$$

The above formula is obtained from Fourier Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

NOTE:- The finite Fourier cosine transform of $f(x)$ in $(0, \pi)$ is defined by

$$F_c(n) = \int_0^{\pi} f(x) \cos nx \, dx$$

where $f(x) = \frac{2}{l} + \sum_{n=1}^{\infty} F_c(n)$

& n is positive integer

NOTE:- The finite Fourier Sine transform of $f(x)$ in $(0, \pi)$ is defined by

$$F_s(n) = \int_0^{\pi} f(x) \sin nx \, dx$$

where $f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{l}\right)$

~~&~~ n is positive integer.

The Finite Fourier Cosine Transform of $f(x)$

The finite Fourier Cosine transform is useful for problems in which the velocity normal to two parallel boundaries are among the boundary conditions

Defination:- The finite Fourier cosine transform of $f(x)$ in $(0, l)$ is defined by

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$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

The above formula is obtained from Fourier Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

NOTE:- The finite Fourier cosine transform of $f(x)$ in $(0, \pi)$ is defined by

$$F_c(n) = \int_0^{\pi} f(x) \cos nx \, dx$$

where $f(x) = \frac{2}{l} + \sum_{n=1}^{\infty} F_c(n)$

& n is positive integer

$\mathscr{D}_\ell$.

Formula.

(1) Finite Fourier Sine transform of $f(x)$ in $(0, l)$

$$F_S(n) = S[f(x)] = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

(2) Finite Fourier Sine transform of $f(x)$ in $(0, \pi)$

$$F_S(n) = S[f(x)] = \int_0^\pi f(x) . \sin nx \, dx$$

(3) Finite Fourier cosine transform of $f(x)$ in $(0, l)$ is $F_C(n) = C[f(x)] = \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

(4) Finite Fourier Cosine transform of $f(x)$ in $(0, \pi)$

$$F_C(n) = C[f(x)] = \int_0^\pi f(x) \cos nx \, dx$$

Fourier Transformation. Examples.

(1) Find the finite Fourier Sine & cosine transformation of $f(x)=1$ in $(0,l)$

Solu^n:- (i) $f(x)=1$ in $(0,l)$, the finite Sine transformation is.

$$F_S(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^l 1\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \left[-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]_0^l = -\left[\frac{l}{2\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= -\left[\frac{l}{2\pi}\cos n\pi\right] = -\left[\frac{l}{2\pi}(-1)^n - \frac{l}{n\pi}\right]$$

$$= -\frac{l}{n\pi}\left[(-1)^n - 1\right]$$

(ii) $f(x)=1$ in $(0,l)$ the finite cosine transformation is

$$F_C(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^l 1\cdot\cos\left(\frac{n\pi x}{l}\right)dx = \left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right]_0^l.$$

$$= \left[\frac{l}{2\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \left[\frac{l}{2\pi}\sin\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right)\right] = \frac{l}{2n\pi}(0) = 0$$

$$F_c(n) = 0.$$

(2) Find the finite Fourier sine transformation of $f(x) = 1$ in $(0, \pi)$.

Solu^n:- $f(x) = 1$ in $(0, \pi)$

The finite Fourier sine transform

$$F_s(n) = \int_0^\pi f(x)\sin\left(\frac{n\pi x}{n}\right)dx$$

$$= \int_0^\pi 1.\sin nx\,dx$$

$$= \left[\frac{-\cos nx}{n}\right]_0^\pi = \left[\frac{-\cos n\pi}{n} + \frac{1}{n}\right]$$

$$F_s(n) = \frac{1}{n}\{1 - (-1)^n\}$$

③ Find Finite Fourier Sine & cosine transformations of $f(x) = x$ in $(0, l)$

Solu^n:- $f(x) = x$ in $(0, l)$

① The finite Fourier Sine transform is

$$F_S(n) = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^l x . \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \left[\frac{-x \cos\left(\frac{n\pi x}{l}\right)}{n\pi/l} + \frac{\sin\left(\frac{n\pi x}{l}\right)}{n^2\pi^2/l^2} - 0 \right]$$

$$= -\frac{l^2}{n\pi} \cos n\pi = -\frac{l^2}{n\pi} (-1)^n.$$

② The finite Fourier cosine transform is

$$F_C(n) = \int_0^l f(x) . \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^l x . \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \left[\frac{x \sin\left(\frac{n\pi x}{l}\right)}{n\pi/l} + \frac{\cos\left(\frac{n\pi x}{l}\right)}{n^2\pi^2/l^2} \right]_0^l$$

$$= \left[0 - 0 + \frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right]\right]$$

$$= \frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right].$$

(2)

(4) Find the finite Fourier Sine & cosine transformations of the fun $f(x) = e^{ax}$ in $(0, l)$

Solu^n:- $f(x) = e^{ax}$ in $(0, l)$

(i) The finite Fourier Sine transform i)

$$F_S(n) = \int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^l e^{ax}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \left[\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left(a.\sin\left(\frac{n\pi x}{l}\right) - \frac{n\pi}{l}.\cos\left(\frac{n\pi x}{l}\right)\right)\right]_0^l$$

$$= \left[\frac{e^{al}\times l^2}{a^2l^2+n^2\pi^2}\left(0 - \frac{\pi n}{l}\cos n\pi\right) - \frac{l^2e^0}{l^2a^2+n^2\pi^2}\left(0 - \frac{n\pi}{l}\right)\right]$$

$$= -\left(\frac{e^{al}.l^2}{l^2a^2+n^2\pi^2}\right)\frac{n\pi}{l}(-1)^n + \frac{l^2}{l^2a^2+n^2\pi^2}\left(\frac{n\pi}{l}\right)$$

$$= \frac{-e^{al}\times ln\pi(-1)^n}{l^2a^2+n^2\pi^2} + \frac{ln\pi}{l^2a^2+n^2\pi^2}$$

$$= \frac{ln\pi}{l^2a^2+n^2\pi^2}\times\left[-e^{al}(-1)^n+1\right]$$

(ii) The finite Fourier Cosine Transform

$$F_c(n) = \int_0^l e^{ax} \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \left[\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(a\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right)\right]_0^l$$

$$= \left\{\frac{l^2 e^{al}}{a^2l^2 + n^2\pi^2}\left(a(-1)^n + 0\right) - \frac{l^2 e^0}{a^2l^2 + n^2\pi^2}(a+0)\right\}$$

$$= \left\{\frac{l^2 e^{al}\left[a \times (-1)^n + 0\right]}{a^2l^2 + n^2\pi^2} - \frac{l^2 e^0}{a^2l^2 + n^2\pi^2}(a+0)\right\}$$

$$= \frac{l^2 e^{al} a(-1)^n}{a^2l^2 + n^2\pi^2} - \frac{l^2 a}{a^2l^2 + n^2\pi^2}$$

$$= \frac{l^2 a}{a^2l^2 + n^2\pi^2}\left[e^{al}(-1)^n - 1\right]$$

5 Find the finite Fourier cosine transformations of $f(x) = 2-x$ $(0,2)$

Solu^n:- $f(x) = 2-x$ over $(0,2)$

The finite Fourier Cosine transform:

$$F_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[(2-x)\left\{\frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2}\right\} - (-1)\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{n^2\pi^2/4}\right\}\right]_0^2$$

$$= \left[\left\{0 - \frac{\cos n\pi}{n^2\pi^2/4}\right\} - \left\{0 - \frac{4}{n^2\pi^2}\right\}\right.$$

$$= \frac{4}{n^2\pi^2}\left\{1-\cos n\pi\right\}$$

$$= \frac{4}{n^2\pi^2}\left\{1-(-1)^n\right\}$$

$=$

⑥ Find Finite Fourier Sine & cosine transformations of $f(x) = (\pi - x)$ in $(0, \pi)$

Solu^n:- $f(x) = \pi - x$ in $(0, \pi)$

(i) The finite Fourier Sine transform is

$$F_S(n) = \int_0^\pi f(x) \sin nx \, dx$$

$$= \int_0^\pi (\pi - x) \sin nx \, dx$$

$$= \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right) \right]_0^\pi$$

$$= \left[(0 + 0) - \left(\frac{\pi}{n} - 0\right) \right]$$

$$F_S(n) = -\frac{\pi}{n}$$

ii) The finite Fourier Cosine transform is

$$F_C(n) = \int_0^\pi f(x) \cos nx \, dx$$

$$= \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right) \right]_0^\pi$$

$$= \left[0 - \frac{\cos n\pi}{n^2} \right] - \left[0 - \frac{1}{n^2} \right] = \frac{1}{n^2} - \frac{\cos n\pi}{n^2}$$

$$= \frac{1}{n^2}(1 - \cos n\pi)$$

$$= \frac{1}{n^2}\left[1 - (-1)^n\right]$$

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(7) Find the finite Fourier Sine transformation of $f(x) = \pi x - x^2$ in $(0, \pi)$

Solu^n:- $f(x) = \pi x - x^2$ in $(0, \pi)$

The finite Fourier Sine Transform is

$$F_s(n) = \int_0^\pi f(x) \sin nx \, dx$$

$$= \int_0^\pi x(\pi - x) \sin nx \, dx$$

$$= \int_0^\pi (\pi x - x^2) \sin nx \, dx$$

$$= \left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right) \right]_0^\pi$$

$$= \left[\left\{ 0 - (\pi - 2\pi)(0) - \frac{2\cos n\pi}{n^2} \right\} - 0 - \pi(0) \right]$$

$$= \frac{-2\cos n\pi}{n^3} + \frac{2}{n^3} = \frac{2}{n^3}\left[1 - \cos n\pi\right]$$

$$= \frac{2}{n^3}\left[1 - (-1)^n\right].$$

8) Find the finite Fourier Cosine Transform of $f(x) = 1+x$ in $(0,3)$

Soln:- $f(x) = 1+x$ in $(0,3)$

The finite Fourier Cosine transform is

$$F_c(n) = \int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^3 (1+x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \left[(1+x)\frac{\sin\left(\frac{n\pi x}{3}\right)}{\frac{n\pi}{3}} - (1)\left(\frac{-\cos\left(\frac{n\pi x}{3}\right)}{\frac{n^2\pi^2}{3^2}}\right)\right]_0^3$$

$$= \left(0 + \frac{9}{n^2\pi^2}\cos n\pi\right) - \left(0 + \frac{9}{n^2\pi^2}\right)$$

$$= \frac{9}{n^2\pi^2}\cos n\pi - \frac{9}{n^2\pi^2}$$

$$= \frac{9}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{9}{n^2\pi^2}\left[(-1)^n - 1\right]$$

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9) Find the finite Fourier sine & cosine transform of $f(x) = x^2$ $(0, 4)$

Soluⁿ:- $f(x) = x^2$ in $(0, 4)$

(i) The finite Fourier sine transform is

$$F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^4 x^2 \cdot \sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \left[\frac{-4}{n\pi} x^2 \cdot \cos\left(\frac{n\pi x}{4}\right) + \frac{8}{n\pi}\left\{\frac{4x}{n\pi}\sin\left(\frac{n\pi x}{4}\right) + \frac{4^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right)\right\}\right]_0^4$$

$$= \frac{-4^3}{n\pi}\cos n\pi + \frac{8}{n\pi}\left[\frac{4x}{n\pi}\sin\left(\frac{n\pi x}{4}\right) + \frac{4^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \frac{-4^3}{n\pi}\cos n\pi + \frac{8}{n\pi}\left(\frac{4^2}{n^2\pi^2}\right)(\cos n\pi - 1)$$

$$= -\frac{64}{n\pi}\cos n\pi + \frac{128}{n^3\pi^3}(\cos n\pi - 1)$$

$$= -\frac{64}{n\pi}(-1)^n + \frac{128}{n^3\pi^3}\left[(-1)^3 - 1\right]$$

(ii) The finite Fourier cosine transform is

$$F_c(n) = \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^4 x^2 \cdot \cos\left(\frac{n\pi x}{4}\right) dx$$

$$= \left[\frac{x^2 \cdot \sin\left(\frac{n\pi x}{4}\right)}{n\pi/4} - 8/n\pi \left(\frac{-x\cos\left(\frac{n\pi x}{4}\right)}{n\pi/4} + \frac{\sin\left(\frac{n\pi x}{4}\right)}{n^2\pi^2/4^2} \right) \right]_0^4$$

$$= \left(\frac{x^2 \sin n\pi}{n\pi/4} \right) - 8/n\pi \left(\frac{-4}{n\pi} \cos \frac{n\pi x}{4} + \frac{4^2}{n^2\pi^2} \sin \frac{n\pi x}{4} \right) \Bigg]_0^4$$

$$= 0 - \frac{128}{n^2\pi^2} \cos(n\pi)$$

$$= \frac{128}{n^2\pi^2}(-1)^n.$$

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8e. (8).

(4) Find the finite Fourier cosine transform of $\left(\frac{\pi - x}{\pi}\right)^2$ in $(0, \pi)$.

Solu^n:- let the given fu^n be

$f(x) = \left(\frac{\pi - x}{\pi}\right)^2$ in $(0, \pi)$.

The finite Fourier cosine transform

$$F_c(n) = \int_0^\pi f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \int_0^\pi \left(1 - \frac{x}{\pi}\right)^2 \cos nx \, dx \quad \text{--- (1)}$$

$$= \left\{ \left[\frac{\sin nx}{n}\left(1 - \frac{x}{\pi}\right)^2\right] - \frac{2}{n\pi}\left[\left(-\frac{\cos nx}{n}\right)\left(1 - \frac{x}{\pi}\right)\right] - (-1)\frac{2}{n^2\pi}\left(1 - \frac{1}{\pi}\right)\left(\frac{\sin nx}{n}\right)\right\}_0^\pi$$

$$= \left\{0 + \frac{2}{n\pi}\left(-1\frac{(-1)^n}{n}\right)\left(1 - \frac{\pi}{\pi}\right)^2 + \frac{2}{n^2\pi}(1) - \frac{1}{\pi}\left(\frac{0}{n}\right)\right\}$$

$$= \left\{\frac{2}{n\pi}\left(\frac{(-1)^{n+1}}{n}\right)(0) + \frac{2}{n^2\pi}(1) - 0\right\}$$

if $n > 0$.

$$= \frac{2}{n^2\pi}$$

8e. (8).

(4) Find the finite Fourier cosine transform of $\left(\frac{\pi-x}{\pi}\right)^2$ in $(0,\pi)$.

Solⁿ:- let the given funⁿ be

$f(x) = \left(\frac{\pi-x}{\pi}\right)^2$ in $(0,\pi)$.

The finite Fourier Cosine Transform

$$F_c(n) = \int_0^\pi f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx \quad \text{---} (1)$$

$$= \int_0^\pi \left(1-\frac{x}{\pi}\right)^2 \cos nx\, dx$$

$$= \left\{\left[\frac{\sin nx}{n}\left(1-\frac{x}{\pi}\right)^2\right] - \frac{2}{n\pi}\left[\left(-\frac{\cos nx}{n}\right)\left(1-\frac{x}{\pi}\right)\right.\right.$$

$$\left.\left. -(-1)\frac{2}{n^2\pi}\left(1-\frac{1}{\pi}\right)\left(\frac{\sin nx}{n}\right)\right]_0^\pi\right.$$

$$= \left\{0 + \frac{2}{n\pi}\left(-1\frac{(-1)^n}{n}\right)\left(1-\frac{\pi}{\pi}\right)^2 + \frac{2}{n^2\pi}(1) - \frac{1}{\pi}\left(\frac{0}{n}\right)\right\}$$

$$= \left\{\frac{2}{n\pi}\left(\frac{(-1)^{n+1}}{n}\right)(0) + \frac{2}{n^2\pi}(1) - 0\right\}$$

if $n > 0$.

$$= \frac{2}{n^2\pi}$$

If $n=0$ Then From (1)

$$F_c(0) = \int_0^{\pi} \left(1-\frac{x}{\pi}\right)^2 (\cos 0.x)\, dx$$

$$= \left[\frac{1}{3}\left(1-x/\pi\right)^3 \left(\frac{1}{-1/\pi}\right)\right]_0^{\pi}$$

$$= -\pi/3 \left[\left(1-x/\pi\right)^3\right]_0^{\pi}$$

$$F_c(0) = \pi/3.$$ //

~~$F_c(n) =$~~

(5) Finite cosine transformation & sine $(x-1)$ in $(0,2)$

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$$F_c(n) = \int_0^2 (x-1)\cos\left(\frac{n\pi x}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$F_s(n) = \int_0^2 (x-1)\sin\left(\frac{n\pi x}{2}\right) dx$$

$$= 2/n\pi \left[1-(-1)^n\right]$$